

Tight Approximation Ratio of Anonymous Pricing

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ABSTRACT

This paper considers two canonical Bayesian mechanism design settings. In the single-item setting, the tight approximation ratio of Anonymous Pricing is obtained: (1) compared to Myerson Auction, Anonymous Pricing always generates at least a $\frac{1}{2.62}$ -fraction of the revenue; (2) there is a matching lower-bound instance.

In the unit-demand single-buyer setting, the tight approximation ratio between the simplest deterministic mechanism and the optimal deterministic mechanism is attained: in terms of revenue, (1) Uniform Pricing admits a 2.62-approximation to Item Pricing; (2) a matching lower-bound instance is presented also.

These results answer two open questions asked by Alaei et al. (FOCS'15) and Cai and Daskalakis (GEB'15). As an implication, in the single-item setting: the approximation ratio of Second-Price Auction with Anonymous Reserve (Hartline and Roughgarden EC'09) is improved to 2.62, which breaks the best known upper bound of $e \approx 2.72$.

CCS CONCEPTS

• **Theory of computation** → **Mathematical optimization; Algorithmic game theory and mechanism design.**

KEYWORDS

optimal auction, anonymous pricing, revenue maximization, mathematical optimization, tight bound

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1 INTRODUCTION

Consider two typical revenue-maximization scenarios: (i) a seller has $n \in \mathbb{N}_{\geq 1}$ items for selling to a unit-demand buyer; and (ii) a seller has a single item for selling to $n \in \mathbb{N}_{\geq 1}$ potential buyers. In the both scenarios, the seller only knows the buyers' value distributions $F = \{F_i\}_{i=1}^n$ for the item(s) instead of their exact values, and wants to maximize his expected revenue.

The simplest mechanisms are to sell the item(s) at a fixed price. In **scenario (i)**, the seller can choose a *uniform price* for all items, and then allows the unit-demand buyer to purchase his favorite item (as long as the value of this item is at least the price). In **scenario (ii)**, the seller can select an *anonymous price* for the item; the first coming buyer (in any order), whose value for the item is at least the price, will take the item.

In practice, the both simple pricing schemes are widely used. In terms of revenue, however, they may not be optimal. In **scenario (i)**, the seller can increase his expected revenue by posting item-wise prices, i.e., the Item Pricing mechanisms in the literature [e.g., see 9, 10, 13–15, 17], which include all deterministic mechanisms¹. In **scenario (ii)**, the seller can even organize an auction, and thus gains more revenue by leveraging the buyer competition. Among these auction schemes, the remarkable Myerson Auction gives the best revenue [see 33].

Compared to the optimal yet complicated mechanisms, how much revenues can the simple and practical pricing schemes guarantee? This is a central question in the theory of Bayesian mechanism design, or more precisely, the “*simple versus optimal*” research program. E.g., as we quote from the survey of Lucier [31]: “*an interesting question is how well one can approximate the optimal revenue using an anonymous price, rather than personalized prices.*”

¹If randomness is allowed, the seller may further increase his revenue by using the Lottery Pricing mechanisms [see 10, 15, 16].

For both of Anonymous Pricing and Uniform Pricing, we settle their approximability in this work.

THEOREM 1 (Myerson Auction vs. Anonymous Pricing). *To sell a single item to multiple buyers with independent regular distributions, the supremum ratio of the Myerson Auction revenue to the Anonymous Pricing revenue equals to the constant*

$$C^* \stackrel{\text{def}}{=} 2 + \int_1^\infty (1 - e^{-Q(x)}) \cdot dx \approx 2.6202,$$

where function $Q(p) \stackrel{\text{def}}{=} -\ln(1 - p^{-2}) - \frac{1}{2} \cdot \sum_{k=1}^\infty k^{-2} \cdot p^{-2k}$.

THEOREM 2 (ITEM PRICING VS. UNIFORM PRICING). *To sell multiple items to a unit-demand buyer with independent regular distributions, the supremum ratio of the Item Pricing revenue to the Uniform Pricing revenue equals to the constant $C^* \approx 2.6202$.*

Notably, the imposed *regularity* assumption (see Section 2 for its definition) of distributions is very standard in microeconomics, and was used in a large volume of previous work. When we allow arbitrarily weird distributions, the both supremum ratios increases to n [see 3, Section 5]. Because those weird distributions are uncommon in practice, the bound of n might not be that informative.

The main body of this work is devoted to establishing the upper-bound part of Theorem 1. For the lower-bound part, Jin et al. [30, Appendix A.4] gave a matching instance.

Chawla et al. [13, 14, 15] proposed the *single-dimensional representative method*², bridging **scenarios (i) and (ii)**. As a result, Theorem 1 implies the upper-bound part of Theorem 2. In addition, after reinterpretation and fine-tuning, the lower-bound instance of Theorem 1 also applies to Theorem 2. (All discussions about Theorem 2 will be presented in the full version of this work.) To the best of our knowledge, Theorem 2 gives the first tight constant approximation ratio in any multi-dimensional setting.

As another implication of Theorem 1, the approximation ratio of Second-Price Auction with Anonymous Reserve [e.g., see 2, 12, 29, 30, 32] against Myerson Auction is improved to $C^* \approx 2.62$. This comparison is a main open problem asked by Hartline and Roughgarden [29], who proved that the tight ratio is between 2 and 4. This range later shrank to $[2.15, 2.72]$ due to Alaei et al. [3] and Jin et al. [30]. Whether our upper bound of $C^* \approx 2.62$ is tight is unknown.

1.1 Our Technique

To prove Theorem 1, like [3, 30], we interpret the ratio as the objective function of a math program, and then manually solve the optimal solution. The variables of this math program are an instance F of the mechanism design problem, i.e., $n \in \mathbb{N}_{\geq 1}$ regular distributions $\{F_i\}_{i=1}^n$ and the number of n itself. Given such a regular instance F : the objective function is the Myerson Auction revenue; the constraint is that the optimal Anonymous Pricing revenue is at most 1.

Formulation. Regarded as functions of the regular instance, the Anonymous Pricing revenue is easy to formulate, whereas the Myerson Auction revenue is quite complex. Given this, Alaei et al. [3]

replaced Myerson Auction with Ex-Ante Relaxation (see [14]), i.e., a “handy” and relaxed objective function. However, the consequent ratio of $e \approx 2.72$ is no longer tight for the Myerson Auction vs. Anonymous Pricing problem.

Conceivably, investigating Myerson Auction itself is necessary for acquiring the exact revenue formula. For this, the main obstacle is that Myerson Auction is clarified based on the *virtual values* [33]. We introduce the virtual value CDF in Section 2, resulting in the desired revenue formula.

We need three different representations of a distribution: **(1)** the CDF F_i is used to formulate the Anonymous Pricing revenue, **(2)** the virtual value CDF D_i is used to formulate the Myerson Auction revenue, and **(3)** the *revenue-quantile curve* r_i is more convenient to decide whether a distribution is regular. In different parts of the proof, we will choose the most suitable representation. This incurs another task – how to bridge these representations. Actually, we observe several one-to-one identities among these representations, which may be of independent interest (cf. the similar structural result in [20, Lemma 5]) and find their applications in the future.

Similar to [3, 30], the proof in this work can be organized into the **Reduction Part** and the **Optimization Part**.

Reduction Part. To solve the Ex-Ante Relaxation vs. Anonymous Pricing problem, Alaei et al. [3] first showed that the worst case is reached by the *triangular distributions*, i.e., a subset of regular distributions. A regular distribution has a *concave-shape* revenue-quantile curve. By contrast, a triangular distribution has a *triangle-shape* revenue-quantile curve, which lies on the boundary of the concave curve family. I.e., a triangular distribution is on the boundary of the regular distribution family. Notably, describing a triangular distribution requires merely two parameters. Thus, the reduction to the triangular distributions greatly simplifies the math program in [3].

Actually, given any regular instance $F = \{F_i\}_{i=1}^n$, the Ex-Ante Relaxation revenue only depends on a specific set of value-probability pairs $\{(b_i, F_i(b_i))\}_{i=1}^n$. Among all regular instances with the same value-probability pairs, there exists a certain instance being stochastically dominated by the others and thus giving the smallest Anonymous Pricing revenue (for any posted price $p \in \mathbb{R}_{\geq 0}$). This special instance is exactly a triangular instance.

In comparison, the Myerson Auction revenue depends on the whole instance $F = \{F_i\}_{i=1}^n$ rather than any set of value-probability pairs. Hence, the mentioned reduction seems inapplicable to the math program in this work. We instead adopt several more complicated reductions, thus to some extent characterizing the worst-case instance of our math program.

More concretely, we observe that a (worst-case) revenue-quantile curve r_i is a triangle or quadrangle with a *curved* edge. Since such a revenue-quantile curve r_i is a concave function (as the distribution F_i is regular), the curved edge is upper-bounded by each tangent line of it; thus, the distribution that corresponds to this tangent line stochastically dominates distribution F_i . Also, the curved edge is lower-bounded by the line connecting its two ends, which corresponds to another distribution that is stochastically dominated by distribution F_i . Each of our reductions never decreases the Myerson Auction revenue. Moreover, with the help of the two auxiliary distributions, we also prove that, after each reduction, the optimal

²The name “single-dimensional representative method” is due to Hartline [28, Chapter 8.5].

Anonymous Pricing revenue is at most 1. I.e., the feasibility of the instance is always guaranteed.

To the best of our knowledge, no reductions tailored specifically to asymmetric regular distributions are previously known. Because many other single-dimensional or multi-dimensional mechanisms in the literature are built on Anonymous Pricing, the techniques in the **Reduction Part** might enlighten the future work on proving tight or tighter approximation ratios of these mechanisms.

Optimization Part. For each math program in [3, 30] and this work, the worst-case instance falls in the family of what we call *continuous instances*. A continuous instance is comprised of infinite buyers, each of which has an infinitesimal buying-probability. Conceivably, the Myerson Auction revenue or the Ex-Ante Relaxation revenue from a continuous instance is captured by an integral. In addition, we will see that the ratio $C^* \approx 2.62$ is exactly the best Myerson Auction revenue achievable by any continuous instance. To settle Theorem 1, it suffices to prove that any feasible instance of our math program can be transformed into another continuous instance, without hurting the Myerson Auction revenue. Such idea was also employed in [3] to obtain the tight ratio of $e \approx 2.72$ for the Ex-Ante Relaxation vs. Anonymous Pricing problem.

As mentioned, the first step in [3] is a reduction from any regular instance to another triangular instance. Actually, any set of triangular distributions intrinsically admits a total order. Also, because of the special structure of Ex-Ante Relaxation, these triangular distributions can be transformed (into a target continuous instance) one by one in the total order. This fact greatly simplifies the proof in [3].

However, a set of regular distributions in general does not admit the mentioned total order. For this reason, in this work: (locally) each distribution has to be transformed piece by piece; (globally) all distributions have to be transformed simultaneously. Besides, Myerson Auction has a more complicated structure than Ex-Ante Relaxation. These issues together incur many technical challenges to implementing the transformation and verifying that the Myerson Auction revenue never decreases.

To enable the proof, *potential function* comes to the rescue: we find a natural potential to indicate the status of an instance. (In some sense, this potential function is a new representation of a distribution.) Given this, we can implement the transformation as an iterative algorithm, during which the potential declines by a fixed amount per iteration. After sufficiently many iterations, the potential ultimately declines to zero, and a desired continuous instance is achieved.

Such transformation is applicable to any instance whose optimal Anonymous Pricing revenue is at most $(1 - \varepsilon)$, i.e., any instance locating in the *interior* of the feasible space of our math program. For any instance locating on the boundary of the feasible space, we still need to convert it into another “interior” instance. This modification may incur a revenue loss of Myerson Auction. However, once the modification is small enough (under some measurement), the revenue loss can be arbitrarily small, which is sufficient to establish Theorem 1.

Even though the ideas of modifying the input instance are widely used in other subareas within TCS (e.g., the smoothed analysis literature), this is the first time that they are used to prove approximation

ratio of simple mechanism. In return, the techniques involved in the **Optimization Part** may even be of interest to the optimization community and the approximation algorithms community.

1.2 Further Related Work

Both of Anonymous Pricing and Uniform Pricing are widely studied in the literature [3, 7–9, 22–24, 28, 30]. In the single-item setting, another important family of pricing schemes is Sequential Posted-Pricing [1, 4, 19, 25, 30, 31]. Such a mechanism allows buyer-wise pricing strategies, and therefore dominates Anonymous Pricing in revenue. Under the regularity assumption, the tight ratio of Sequential Posted-Pricing to Anonymous Pricing also equals to $C^* \approx 2.62$ [30].

In the single-buyer unit-demand setting, the family of Item Pricing mechanisms includes all deterministic mechanisms, among which finding the optimum is NP-hard [17]. If randomness is allowed, the seller can gain more revenue by employing *lottery* [10, 15, 16, 27]. Chen et al. [16] settled the complexities of finding and describing the optimal randomized mechanism.

Other more general multi-item settings involve single or multiple buyers with unit-demand or other utility functions, in which optimal mechanisms can be much more complex. For this reason, the last two decades have seen a great deal of work on proving the intractability of optimal mechanisms, and an richer literature on proving that simple mechanisms approximate optimal mechanisms by constant factors. In this amount of space, evaluating so extensive a literature is impossible. As a guideline, the reader can refer to the hardness results in [5, 16–18, 21, 26, 34], the approximation results in [6, 9–11, 13–15, 35], and the references therein.

2 NOTATION AND PRELIMINARIES

Below, we formally define in the mathematical notions to be used in the paper.

- Function $(\cdot)_+$ maps any real number $z \in \mathbb{R}$ to $\max\{0, z\}$.
- Function $\mathbb{1}\{\cdot\}$ denotes the indicator function.
- $g(z^-) \stackrel{\text{def}}{=} \lim_{w \rightarrow z^-} g(w)$ and $g(z^+) \stackrel{\text{def}}{=} \lim_{w \rightarrow z^+} g(w)$ respectively denote the left and the right limitations (if exist) of a function g in the neighborhood of $w = z$.
- $\partial_- g(z) \stackrel{\text{def}}{=} \lim_{w \rightarrow z^-} \frac{g(w) - g(z)}{w - z}$ and $\partial_+ g(z) \stackrel{\text{def}}{=} \lim_{w \rightarrow z^+} \frac{g(w) - g(z)}{w - z}$ respectively denote the left derivative and the right derivative (if exist) of a function g at $w = z$.
- For any increasing function g (may not be strictly increasing), define its inverse function as $g^{-1}(z) \stackrel{\text{def}}{=} \max\{w \in \mathbb{R} \mid g(w) \leq z\}$. Similarly, for any decreasing function g , define its inverse function as $g^{-1}(z) \stackrel{\text{def}}{=} \min\{w \in \mathbb{R} \mid g(w) \geq z\}$.

Probability Distribution. We use three mathematically equivalent representations to describe a regular distribution: cumulative distribution function, revenue-quantile curve, and virtual value cumulative distribution function. Actually, the first and the second representations (to be introduced in Section 2.1) are applicable to more general distributions.

2.1 Cumulative Distribution Function and Revenue-Quantile Curve

The most natural way to describe a distribution is the cumulative distribution function (CDF) F_i , which is assumed to be a left-continuous function for convenience. I.e., let b_i be a random variable drawn from this distribution, then $F_i(x) = \Pr\{b_i < x\}$ for all $x \in \mathbb{R}$. When there is no ambiguity, we also denote by F_i the distribution.

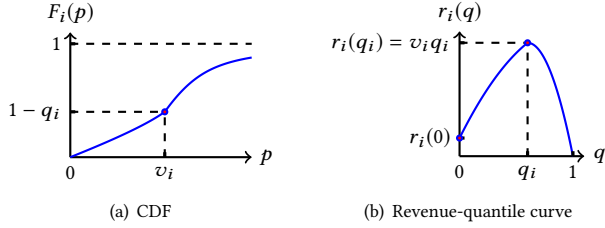


Figure 1: Demonstration for the CDF and the revenue-quantile curve.

Revenue-Quantile Curve. If distribution F_i has a non-negative support $\text{supp}(F_i) \subseteq \mathbb{R}_{\geq 0}$, the revenue-quantile curve r_i defined below also records all details, thus being a new representation. The both representations are respectively illustrated in Figures 1(a) and 1(b).

FACT 1. For any distribution, its CDF F_i and revenue-quantile curve r_i admit the reductions below:

- (1) $r_i(q) = q \cdot F_i^{-1}(1 - q)$ for all $q \in (0, 1]$. In addition, $r_i(0) = \lim_{q \rightarrow 0^+} r_i(q) = \lim_{p \rightarrow \infty} p \cdot (1 - F_i(p))$ if this limitation exists. (r_i is a left-continuous function à la F_i)
- (2) $F_i(p) = \min_{q \in [0, 1]} \{1 - q \mid r_i(q)/q \geq p\}$ for all $p \in \mathbb{R}_{\geq 0}$.

Extra Notation. We now define several useful quantities based on the CDF F_i :

- The *support-supremum* $u_i \stackrel{\text{def}}{=} \max \{\text{supp}(F_i)\} \in \mathbb{R}_{\geq 0}$.
- The *monopoly price* $v_i \in \text{argmax}_{p \in \mathbb{R}_{\geq 0}} \{p \cdot (1 - F_i(p))\}$. When there are multiple alternative monopoly prices, we break ties by choosing the largest one.
- The *monopoly quantile* $q_i \stackrel{\text{def}}{=} (1 - F_i(v_i)) \in [0, 1]$.

Alternatively, we can defined these quantities based on the revenue-quantile curve r_i as well:

- The *support-supremum* $u_i = \lim_{q \rightarrow 0^+} r_i(q)/q$.
- The *monopoly quantile* $q_i \in \text{argmax}_{q \in [0, 1]} \{r_i(q)\}$. When there are multiple alternative monopoly quantiles, we break ties by choosing the smallest one.
- The *monopoly price* $v_i = r_i(q_i)/q_i$.

Note that $v_i \leq u_i$ and possibly $v_i = u_i = \infty$. To comprehend these notions intuitively, consider this pricing scenario: a seller wants to sell a single item by posting a price of $p \in \mathbb{R}_{\geq 0}$; a single buyer draws his value of $b_i \in \mathbb{R}_{\geq 0}$ from a distribution F_i , and takes the item iff his value is at least the posted price of p_i . Clearly, the value of b_i is capped to the support-supremum u_i . Also, the seller can maximize his expected revenue by posting the monopoly price

$p = v_i$, resulting in a selling probability (i.e., the quantile) of the monopoly quantile q_i .

2.2 Regular Distribution and Virtual Value

We denote by REG the set of regular distributions. It is well known [e.g., see 33] that there are several equivalent ways to describe such a distribution. Among these equivalent definitions, we will choose the most convenient one in different parts of this work.

DEFINITION 1 (REGULAR DISTRIBUTION). The following conditions for the regularity are equivalent.

- (1) The virtual value CDF D_i is well defined, and has a finite expectation $\int_{\mathbb{R}} z \cdot dD_i(z)$.
- (2) The revenue-quantile curve r_i is a continuous and concave function on interval $q \in [0, 1]$.
- (3) The virtual value function $\varphi_i(p) = p - \frac{1 - F_i(p)}{f_i(p)}$ is an increasing function on interval $p \in \mathbb{R}_{\geq 0}$ (with the conventions that $\frac{0}{0} = 0$, $\frac{z}{0} = \infty$ and $\frac{z}{\infty} = 0$ when $z \in \mathbb{R}_{\geq 0}$), where $f_i(p) \stackrel{\text{def}}{=} \partial_- F_i(p)$ is the probability density function (PDF).

Together with a positive constant $r_i(0) \geq (-\int_{\mathbb{R}} z \cdot dD_i(z))_+$, we can reconstruct the CDF F_i and the revenue-quantile curve r_i from the virtual value CDF D_i and vice versa.

FACT 2. For any regular distribution $F_i \in \text{REG}$, its virtual value CDF D_i (well defined) and revenue-quantile curve r_i (continuous and concave) admit the reductions below:

- (1) $r_i(q) = r_i(0) + \int_0^q D_i^{-1}(1 - z) \cdot dz$ for all $q \in [0, 1]$.
- (2) $D_i(x) = \min_{q \in [0, 1]} \{1 - q \mid \partial_- r_i(q) \geq x\}$ for all $x \in \mathbb{R}$, with the convention $\partial_- r_i(0) = \infty$.

FACT 3. For any regular distribution $F_i \in \text{REG}$, its virtual value CDF D_i (well defined) and CDF F_i (with an increasing virtual value function φ_i) admit the reductions below:

- (1) $\varphi_i(F_i^{-1}(1 - q)) = D_i^{-1}(1 - q)$ for all $q \in [0, 1]$.
- (2) $D_i(x) = F_i(\varphi_i^{-1}(x^-))$ for all $x \in \mathbb{R}$.

By the above discussions, we can infer the next Corollary 1.

COROLLARY 1. For any regular distribution $F_i \in \text{REG}$, the following holds.

- (1) In interval $q \in [0, 1]$, the revenue-quantile curve r_i is left- and right-differentiable everywhere. At any quantile $q \in [0, 1]$, the left derivative equals to the corresponding virtual value, i.e., $\partial_- r_i(q) = D_i^{-1}(1 - q) = \varphi_i(F_i^{-1}(1 - q))$.
- (2) Only at the support-supremum $u_i = \max \{\text{supp}(F_i)\} \in \mathbb{R}_{\geq 0}$ the distribution F_i may have a probability mass; the CDF F_i is left- and right-differentiable anywhere else. W.l.o.g., the PDF f_i exists and is left-continuous.
- (3) The CDF stochastically dominates the virtual value CDF, i.e., $F_i(x) \leq D_i(x)$ for all $x \in \mathbb{R}$.

As the next Figure 2 shows, the quantities defined before also have geometric meanings w.r.t. the virtual value CDF:

- The support-supremum $u_i = \max \{\text{supp}(D_i)\} = \varphi_i(u_i^+)$ when the constant $r_i(0) = 0$ and $u_i = \infty$ when the constant $r_i(0) > 0$.
- The monopoly quantile $q_i = 1 - D_i(0)$.

- The monopoly price $v_i = q_i^{-1} \cdot (r_i(0) + \int_{\mathbb{R}_{\geq 0}} x \cdot dD_i(x))$, giving a virtual value of $\varphi_i(v_i^+) = \min \{\text{supp}(D_i) \cap \mathbb{R}_{\geq 0}\}$.

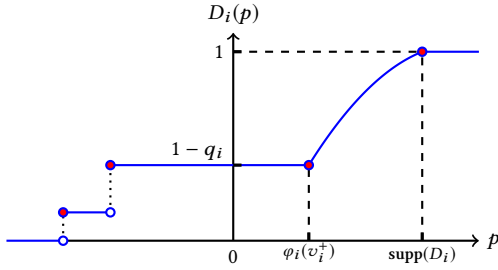


Figure 2: Demonstration for the virtual value CDF of a regular distribution.

Triangular Distribution. This family of distributions [denoted by TRI; introduced by 3] is a subset of the regular distribution family REG. Such a distribution has the same support-supremum and monopoly price $u_i = v_i \in \mathbb{R}_{\geq 0}$. Given the monopoly quantile $q_i \in [0, 1]$, a triangular distribution $\text{TRI}(v_i, q_i)$ can be represented as follows. As Figure 3(b) below shows, the triangular distribution $\text{TRI}(v_i, q_i)$ has a triangle-shape revenue-quantile curve r_i .

- The CDF $F_i(p) = \frac{p \cdot (1 - q_i)}{p \cdot (1 - q_i) + v_i q_i}$ for all $p \in [0, v_i]$ and $F_i(p) = 1$ for all $p \in (v_i, \infty)$.
- The revenue-quantile curve $r_i(q) = v_i \cdot q$ for all $q \in [0, q_i]$ and $r_i(q) = \frac{v_i q_i}{1 - q_i} \cdot (1 - q)$ for all $q \in [q_i, 1]$.
- The virtual value CDF $D_i(x) = (1 - q_i) \cdot \mathbb{1}\{x > \frac{v_i q_i}{1 - q_i}\} + q_i \cdot \mathbb{1}\{x > v_i\}$ for all $x \in \mathbb{R}$.

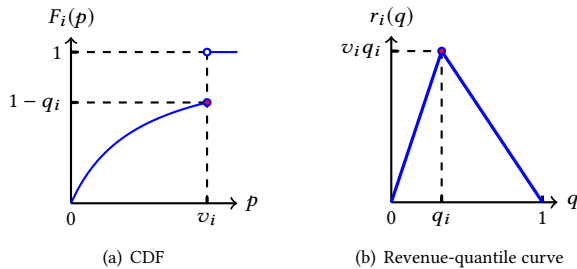


Figure 3: Demonstration for distribution $\text{TRI}(v_i, q_i)$

Particularly, we denote by $\text{TRI}(\infty)$ the limitation distribution $\lim_{N \rightarrow \infty} \text{TRI}(N, 1/N)$, which has a CDF of $F(p) = \frac{p}{p+1}$ for all $p \in \mathbb{R}_{\geq 0}$ and a revenue-quantile curve of $r(q) = 1 - q$ for all $q \in [0, 1]$. We will see that the worst-case instance of the Myerson Auction vs. Anonymous Pricing problem involves this distribution $\text{TRI}(\infty)$.

2.3 Myerson Auction and Anonymous Pricing

In the single-item Bayesian mechanism design setting, a seller wants to sell an indivisible item to $n \in \mathbb{N}_{\geq 1}$ potential buyers. The buyers independently draw their values $\{b_i\}_{i=1}^n \in \mathbb{R}_{\geq 0}^n$ from the distributions $\{F_i\}_{i=1}^n$.

Myerson Auction. For any regular³ instance $\mathbf{F} = \{F_i\}_{i=1}^n \in \text{REG}^n$, upon receiving the values $\{b_i\}_{i=1}^n$ from the buyers, the seller performs Myerson Auction as follows:

- (1) For some permutation $\{\sigma_i\}_{i=1}^n \in \Pi$, the virtual values $\varphi_{\sigma_1}(b_{\sigma_1}) \geq \varphi_{\sigma_2}(b_{\sigma_2}) \geq \dots \geq \varphi_{\sigma_{n-1}}(b_{\sigma_{n-1}}) \geq \varphi_{\sigma_n}(b_{\sigma_n})$
- (2) If the highest virtual value $\varphi_{\sigma_1}(b_{\sigma_1}) \geq 0$, then buyer σ_1 wins the item, with a payment of $\varphi_{\sigma_1}^{-1}(\max\{0, \varphi_{\sigma_2}(b_{\sigma_2})\})$

We denote by $\text{OPT}(\mathbf{F})$ or simply OPT (if the instance $\mathbf{F} = \{F_i\}_{i=1}^n$ is understood) the Myerson Auction revenue, which is formulated as Fact 4 in the below.

FACT 4 (MYERSON AUCTION REVENUE FORMULA). For any regular instance $\mathbf{F} = \{F_i\}_{i=1}^n \in \text{REG}^n$,

$$\text{OPT}(\mathbf{F}) = \sum_{i=1}^n r_i(0) + \int_0^\infty \left(1 - \prod_{i=1}^n D_i(x)\right) \cdot dx.$$

Anonymous Pricing. In such a mechanism, the seller posts a price of $p \in \mathbb{R}_{\geq 0}$ to the item. The item is sold out iff at least one buyer $i \in [n]$ has a value of $b_i \geq p$. Obviously, the selling probability equals to $(1 - \prod_{i=1}^n F_i(p))$, resulting in an expected revenue of

$$\text{AP}(p, \mathbf{F}) \stackrel{\text{def}}{=} p \cdot \left(1 - \prod_{i=1}^n F_i(p)\right).$$

Again, we write $\text{AP}(p)$ instead of $\text{AP}(p, \mathbf{F})$ when the instance $\mathbf{F} = \{F_i\}_{i=1}^n$ is understood. The seller will choose an optimal posted price, hence a revenue of $\text{AP} \stackrel{\text{def}}{=} \max_{p \in \mathbb{R}_{\geq 0}} \{\text{AP}(p)\}$.

The Myerson Auction vs. Anonymous Pricing problem can be formulated the following Program (P0). Notice that constraint (C0) trivially holds when $p \in (0, 1)$.

$$\begin{aligned} \max_{\mathbf{F} \in \text{REG}^n} \quad & \text{OPT} = \sum_{i=1}^n r_i(0) + \int_0^\infty \left(1 - \prod_{i=1}^n D_i(x)\right) \cdot dx \quad (\text{P0}) \\ \text{s.t.} \quad & \text{AP}(p) = p \cdot \left(1 - \prod_{i=1}^n F_i(p)\right) \leq 1, \forall p \in [1, \infty] \quad (\text{C0}) \end{aligned}$$

3 PROOF OVERVIEW

The proof of Theorem 1 is enabled by finding the worst-case instance of Program (P0). We will outline the approach in this section (with more details postponed to the full version of this work), which can be divided into the following two parts.

Reduction Part. In Sections 3.1 to 3.3, we will present several reductions among the feasible instances of Program (P0), with the purpose of characterizing the worst case. As a result, the optimal objective value of Program (P0) is upper-bounded by that of the following Program (P1).

³For an irregular instance, Myerson Auction is more complex and may be randomized. Also, the revenue formula in Fact 4 should take into account the virtual value CDF's after the *ironing* process [33].

$$\begin{aligned}
\max_{F \in \text{REG}^n} \quad & \text{OPT} = 2 + \int_1^\infty \left(1 - \prod_{i=1}^n D_i(x)\right) \cdot dx & (\text{P1}) \\
\text{s.t.} \quad & \sum_{i=1}^n \ln \left(1 + p \cdot \frac{1 - F_i(p)}{F_i(p)}\right) \leq -p \cdot \ln(1 - p^{-2}), \\
& \forall p \in [1, \infty] & (\text{C1}) \\
& r_i(q) = \frac{r_i(q_i)}{1 - q_i} \cdot (1 - q) \text{ for all } q \in [q_i, 1], \\
& \forall i \in [n] & (\text{C2}) \\
& v_i > 1, \varphi_i(v_i^+) \geq 1, 0 < q_i < r_i(q_i) \leq 1, r_i(0) = 0, \\
& \forall i \in [n] & (\text{C3})
\end{aligned}$$

Optimization Part. We will define a special feasible instance of Program (P1) in Section 3.4, and then prove that this instance is actually a worst-case instance (see Sections 3.5 to 3.7). To convey the main ideas behind the proof, a “proof plan” is provided at the end of Section 3.4.

3.1 A Special Buyer $\text{TRI}(1, 1)$

Given any feasible instance $F = \{F_i\}_{i=1}^n$ of Program (P0), we investigate the composition of it with the special triangular distribution $\text{TRI}(1, 1)$. By definition (see Section 2.1), the triangular distribution $\text{TRI}(1, 1)$ has the same CDF and virtual value CDF $F_0(p) = D_0(p) = \mathbb{1}\{p > 1\}$.

First, the composition instance is feasible to Program (P0) as well. I.e., for any posted price of $p \in [1, \infty]$, since $F_0(p) = 1$, the composition instance gives the same Anonymous Pricing revenue as the given instance F. Second, compared to the given instance F, the composition instance $\text{TRI}(1, 1) \cup F$ gives a higher Myerson Auction revenue. In that $D_0(p) = \mathbb{1}\{p > 1\}$,

$$\text{OPT}(\text{TRI}(1, 1) \cup F) = 1 + \sum_{i=1}^n r_i(0) + \int_1^\infty \left(1 - \prod_{i=1}^n D_i(x)\right) \cdot dx.$$

This modified objective function indicates that *any other buyer now contributes to the Myerson Auction revenue only when his virtual value is higher than 1*. This observation enables the next Main Lemma 1. To establish this main lemma, we employ the ideas involved in [3, Lemma 3.1].

MAIN LEMMA 1. *For any worst-case instance $\{\text{TRI}(1, 1)\} \cup F$ of Program (P0), w.l.o.g. the following holds for each $i \in [n]$:*

- (1) *The revenue-quantile curve $r_i(q) = \frac{r_i(q_i)}{1 - q_i} \cdot (1 - q)$ for all $q \in [q_i, 1]$.*
- (2) *If the monopoly quantile $q_i > 0$, then $\partial_- r_i(q) > 1$ for all $q \in [0, q_i]$.*

We transform Program (P0) into the next Program (P0'). Clearly, constraint (C1') is due to Main Lemma 1.1. For constraint (C2'), the first and the second inequalities are due to Main Lemma 1.2: the monopoly price $v_i = \frac{r_i(q_i)}{q_i} > 1$ (with the convention that $\frac{z}{0} = \infty$ when $z > 0$) and the monopoly virtual value $\varphi_i(v_i^+) = \lim_{q \rightarrow q_i^-} \partial_- r_i(q) \geq 1$. Since $v_i > 1$, the revenue-quantile curve $r_i(q_i) = v_i q_i > q_i \geq 0$. In addition, $1 \geq \text{AP}(v_i) \geq v_i \cdot (1 - F_i(v_i)) = r_i(q_i)$, where the first inequality is due to constraint (C0'). For

ease of notation, we never explicitly mention the special triangular distribution $\text{TRI}(1, 1)$ elsewhere.

$$\begin{aligned}
\max_{F \in \text{REG}^n} \quad & \text{OPT} = 1 + \sum_{i=1}^n r_i(0) + \int_1^\infty \left(1 - \prod_{i=1}^n D_i(x)\right) \cdot dx & (\text{P0}') \\
\text{s.t.} \quad & \text{AP}(p) = p \cdot \left(1 - \prod_{i=1}^n F_i(p)\right) \leq 1, \quad \forall p \in [1, \infty] & (\text{C0}') \\
& r_i(q) = \frac{r_i(q_i)}{1 - q_i} \cdot (1 - q) \text{ for all } q \in [q_i, 1], \\
& \forall i \in [n] & (\text{C1}') \\
& v_i > 1, \varphi_i(v_i^+) \geq 1, 0 < q_i < r_i(q_i) \leq 1, \\
& \forall i \in [n] & (\text{C2}')
\end{aligned}$$

3.2 A Special Buyer $\text{TRI}(\infty)$

We further analyze the worst-case instance of Program (P0'), resulting in the next Main Lemma 2.

MAIN LEMMA 2 (A SPECIAL BUYER $\text{TRI}(\infty)$). *For any worst-case instance $F = \{F_i\}_{i=0}^n$ of Program (P0'), w.l.o.g. the following holds:*

- (1) *$r_0(q) = 1 - q$ for all $q \in [0, 1]$, i.e., $F_0(p) = \frac{p}{p+1}$ for all $p \in \mathbb{R}_{\geq 0}$.*
- (2) *$q_i > 0$ and $r_i(0) = 0$ for each $i \in [n]$.*

Based on Main Lemma 2, we can transform Program (P0') into the next Program (P0''). Concretely, objective (P0'') is due to objective (P0'), by taking into account Main Lemma 2 as (in a worst case) $\sum_{i=0}^n r_i(0) = 1 + \sum_{i=1}^n r_i(0) = 1$. Besides, constraint (C2'') is due to constraint (C2') and Main Lemma 2.2. For simplicity, we never explicitly mention the special distribution $\text{TRI}(\infty)$ elsewhere.

$$\begin{aligned}
\max_{F \in \text{REG}^n} \quad & \text{OPT} = 2 + \int_1^\infty \left(1 - \prod_{i=1}^n D_i(x)\right) \cdot dx & (\text{P0}'') \\
\text{s.t.} \quad & \text{AP}(p) = p \cdot \left(1 - \frac{p}{p+1} \cdot \prod_{i=1}^n F_i(p)\right) \leq 1, \\
& \forall p \in [1, \infty] & (\text{C0}'') \\
& r_i(q) = \frac{r_i(q_i)}{1 - q_i} \cdot (1 - q) \text{ for all } q \in [q_i, 1], \\
& \forall i \in [n] & (\text{C1}'') \\
& v_i > 1, \varphi_i(v_i^+) \geq 1, 0 < q_i < r_i(q_i) \leq 1, r_i(0) = 0, \\
& \forall i \in [n] & (\text{C2}'')
\end{aligned}$$

3.3 A Simple Relaxation

Comparing Program (P0'') to the desired Program (P1), we notice that only constraint (C0'') differs from constraint (C1). To complete the transformation, we rearrange constraint (C0''), hence another mathematically equivalent constraint (C0''').

$$\begin{aligned}
& \sum_{i=1}^n \ln \left(1 + \frac{1 - F_i(p)}{F_i(p)}\right) \leq -\ln(1 - p^{-2}); & (\text{C0}''') \\
& \sum_{i=1}^n \ln \left(1 + p \cdot \frac{1 - F_i(p)}{F_i(p)}\right) \leq -p \cdot \ln(1 - p^{-2}). & (\text{C1})
\end{aligned}$$

Clearly, constraint (C1) can be derived from constraint (C0''') via a standard trick [e.g., see 3, Lemma 3.4], i.e., $\ln(1+z) \geq p^{-1} \cdot \ln(1+p \cdot z)$ when $p > 1$ and $z \geq 0$.

This accomplishes the **Reduction Part** of Section 3. To see Theorem 1, it suffices to prove that the optimal objective value of Program (P1) is upper-bounded by the constant $C^* \approx 2.62$.

3.4 Continuous Instance

In the reminder of Section 3 (i.e., the **Optimization Part**), we will grasp a specific worst-case instance of Program (P1). This instance lies in the family of what we call *continuous instances*. Parameterized by $\gamma \in [1, \infty]$, a continuous instance $\text{CONT}(\gamma)$ is comprised of a spectrum of “small” triangular distributions, making constraint (C1) tight everywhere in interval $p \in [\gamma, \infty]$. Denote by function $\mathcal{R}(p) \stackrel{\text{def}}{=} -p \cdot \ln(1 - p^{-2})$ the RHS of constraint (C1). Based on function \mathcal{R} , we provide a formal definition of the continuous instance $\text{CONT}(\gamma)$ below.

DEFINITION 2. Given any parameter $\gamma \in [1, \infty]$, for any positive integer $m \in \mathbb{N}_{\geq 1}$, consider the following triangular instance $\{\text{TRI}(v_i, q_i)\}_{i=1}^{m^2}$:

- Let $v_i \stackrel{\text{def}}{=} \gamma + m - \frac{i-1}{m}$ for each $i \in [m^2]$. For ease of notation, let $v_0 \stackrel{\text{def}}{=} \infty$.
- Let $\sum_{j=1}^i \ln(1 + \frac{v_j q_j}{1 - q_j}) = \mathcal{R}(v_i)$, i.e., $q_i \stackrel{\text{def}}{=} \frac{e^{\mathcal{R}(v_i) - \mathcal{R}(v_{i-1})} - 1}{v_i + e^{\mathcal{R}(v_i) - \mathcal{R}(v_{i-1})} - 1}$ for each $i \in [m^2]$.

Then, the continuous instance $\text{CONT}(\gamma)$ is defined as the limitation instance $\lim_{m \rightarrow \infty} \{\text{TRI}(v_i, q_i)\}_{i=1}^{m^2}$. By construction, each triangular instance $\{\text{TRI}(v_i, q_i)\}_{i=1}^{m^2}$ and the continuous instance $\text{CONT}(\gamma)$ are feasible to Program (P1).

Recall function $Q(p) = -\ln(1 - p^{-2}) - \frac{1}{2} \cdot \sum_{k=1}^{\infty} k^{-2} \cdot p^{-2k}$ involved in Theorem 1. In the next Fact 5, we get a Myerson Auction revenue formula tailored specifically to continuous instances. The subsequent Figure 4 is offered for demonstration.

FACT 5 (MYERSON AUCTION REVENUE FORMULA FOR CONTINUOUS INSTANCE). Given any continuous instance $\text{CONT}(\gamma)$, where $\gamma \in [1, \infty]$, the Myerson Auction revenue equals to

$$\text{OPT}(\text{CONT}(\gamma)) = 2 + \int_1^{\infty} (1 - e^{-Q(\max\{x, \gamma\})}) \cdot dx,$$

which is a decreasing function on interval $\gamma \in [1, \infty]$.

The reader may wonder how to get Fact 5 from Definition 2: the revenue formula involves function Q , whereas a continuous instance $\text{CONT}(\gamma)$ is defined based on function \mathcal{R} . Actually, there is a *differential equation* bridging the both functions and thus enabling Fact 5. (We defer such technical details to the full version.) It is noteworthy that, among all continuous instances, $\text{CONT}(1)$ gives the best Myerson Auction revenue, i.e.,

$$\text{OPT}(\text{CONT}(1)) = C^* = 2 + \int_1^{\infty} (1 - e^{-Q(x)}) \cdot dx \approx 2.62.$$

Proof Plan and Organization. In Sections 3.5 to 3.7, we will prove that the special continuous instance $\text{CONT}(1)$ is a worst-case instance of Program (P1).

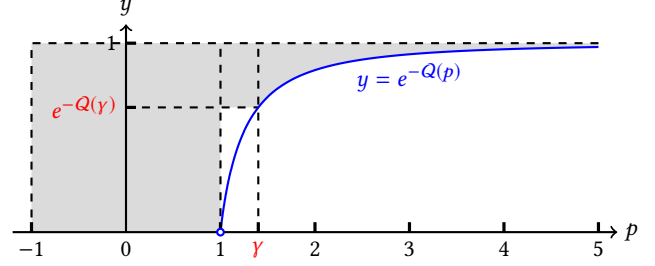


Figure 4: Demonstration for function $y = e^{-Q(p)}$; the shadow area equals to the Myerson Auction revenue $\text{OPT}(\text{CONT}(\gamma))$, which is a decreasing function.

To this end, we will convert any feasible instance $F = \{F_i\}_{i=1}^n$ of Program (P1) to a targeted continuous instance $\text{CONT}(\gamma^*)$, without hurting the Myerson Auction revenue. More concretely:

- We construct a finite-length **LIST** of instances, beginning with the discrete instance F (also denoted by **GIVEN**), and ending up with the targeted continuous instance $\text{CONT}(\gamma^*)$ (also denoted by **TAIL**). In the interior of the **LIST**, any other instance is composed of a discrete component and a continuous component.
- Consider a pair of consecutive instances **PREV** and **NEXT** in the **LIST**. The instance **NEXT** is constructed from the instance **PREV** in a consistent pattern: we always “diminish” the discrete component and “augment” the continuous component.
- The construction of the whole **LIST** will be implemented as an iterative algorithm.

The remainder of Section 3 is organized as follows:

- In Section 3.5, we first clarify how to construct a single distribution in each iteration, and then prove that this construction preserves the feasibility to constraints (C2) and (C3).
- In Section 3.6, we will elaborate on the construction in each iteration (i.e., how to obtain an instance **NEXT** from an instance **PREV**), and afterward validate the feasibility to constraints (C1) under this construction.
- In Section 3.7, we will implement the iterative algorithm based on the proposed constructions, showing that the targeted continuous instance **TAIL** = $\text{CONT}(\gamma^*)$ brings a higher Myerson Auction revenue than the discrete instance $F = \text{GIVEN}$. As mentioned in Section 3.4, among all continuous instances, $\text{CONT}(1)$ gives the best Myerson Auction revenue of the constant $C^* \approx 2.62$. Combining everything together accomplishes the proof of Theorem 1.

3.5 Construction of a New Distribution \bar{F}_k

To describe our construction of a new distribution \bar{F}_k from a given feasible distribution F_k of Program (P1), we need the following requisite notions about *potential*.

DEFINITION 3 (POTENTIAL OF A DISTRIBUTION). For any regular distribution $F_k \in \text{REG}$:

- Define the potential function $\Psi_k(p) \stackrel{\text{def}}{=} \ln\left(1 + p \cdot \frac{1 - F_k(p)}{F_k(p)}\right)$ on interval $p \in \mathbb{R}_{\geq 0}$.
- Define the gross potential $\Psi_k \stackrel{\text{def}}{=} \Psi_k(v_k) = \ln\left(1 + \frac{v_k q_k}{1 - q_k}\right)$.

It is noteworthy that the potential function $\Psi_k(p)$ is exactly the k -th summand on the LHS of constraint (C1). Clearly, this function records all details about distribution F_k , thus serving as a new representation of distribution F_k . In the next Fact 6, we study the monotonicity of the potential function $\Psi_k(p)$, which is useful for our later proofs.

FACT 6 (POTENTIAL OF A DISTRIBUTION). For any feasible distribution F_k of Program (P1):

- (1) The potential function $\Psi_k(p)$ is a decreasing function on interval $p \in \mathbb{R}_{\geq 0}$.
- (2) $\Psi_k(p) = \Psi_k$ when $p \in [0, v_k]$.
- (3) $\Psi_k(p) = 0$ when $p \in (u_k, \infty)$.

3.5.1 The Potential-Based Construction DIMINISH. Given any distribution F_k feasible to Program (P1), our construction of a new distribution \bar{F}_k is controlled by a *pointwise* potential-decrease of $\Delta_k \in [0, \Psi_k]$. As the following Figure 5 shows, let $\bar{\Psi}_k(p) \stackrel{\text{def}}{=} \ln\left(1 + p \cdot \frac{1 - \bar{F}_k(p)}{\bar{F}_k(p)}\right)$ denote the new potential function,

$$\bar{\Psi}_k(p) = (\Psi_k(p) - \Delta_k)_+, \quad \forall p \in \mathbb{R}_{\geq 0},$$

where function $(\cdot)_+$ maps a real number $z \in \mathbb{R}$ to $\max\{z, 0\}$. The potential-decrease Δ_k is capped to the gross potential $\Psi_k = \ln\left(1 + \frac{v_k q_k}{1 - q_k}\right)$. Particularly, if $\Delta_k = \Psi_k$, then $\bar{F}_k(p) = 1$ for all $p \in \mathbb{R}_{\geq 0}$. We call the construction “**DIMINISH** the given distribution F_k by a factor of Δ_k ” or simply $\bar{F}_k \leftarrow \text{DIMINISH}(F_k, \Delta_k)$.

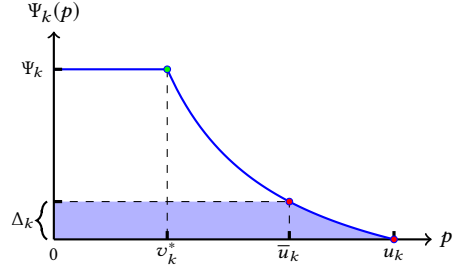
We now review several notions tailored specifically to the new distribution \bar{F}_k .

- The monopoly price $\bar{v}_k = v_k$, which can be inferred from Fact 6 and Figure 5. To emphasize that the monopoly price is invariant, we mark it with an asterisk*.
- The monopoly quantile $\bar{q}_k \leq q_k$. By construction, we have $\ln\left(1 + \frac{v_k^* \bar{q}_k}{1 - \bar{q}_k}\right) = \ln\left(1 + \frac{v_k^* q_k}{1 - q_k}\right) - \Delta_k$.
- The support-supremum $\bar{u}_k = \Psi_k^{-1}(\Delta_k) \in [v_k^*, u_k]$, as Figure 5 suggests.
- W.l.o.g., both of the new CDF \bar{F}_k and the new PDF \bar{f}_k are left-continuous.

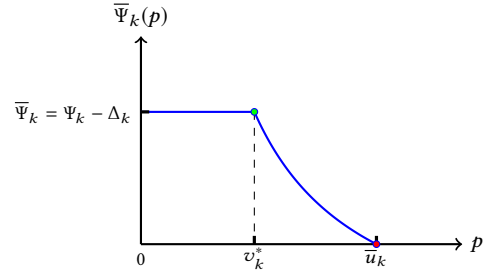
Later, we will prove that the new distribution \bar{F}_k is regular (see Main Lemma 3 in Section 3.5.2), which implies the following:

- The virtual value function $\bar{\varphi}_k(p) \stackrel{\text{def}}{=} p - \frac{1 - \bar{F}_k(p)}{\bar{f}_k(p)}$ is an increasing function on interval $p \in \mathbb{R}_{\geq 0}$.
- The revenue-quantile curve \bar{r}_k is a concave function on interval $q \in [0, 1]$. By construction, we also have $\bar{r}_k(q) \leq r_k(q)$ for all $q \in [0, 1]$.
- The virtual value CDF \bar{D}_k is well defined.

3.5.2 Properties of the Construction DIMINISH. In this part, we present several properties of the construction DIMINISH, which are cornerstones of our later proofs. First, as Figure 5(b) shows, Fact 6 also holds for the new distribution \bar{F}_k :



(a) Potential function of distribution F_k



(b) Potential function of distribution \bar{F}_k

Figure 5: Demonstration for the construction of \bar{F}_k .

- (1) $\bar{\Psi}_k(p) \stackrel{\text{def}}{=} \ln\left(1 + p \cdot \frac{1 - \bar{F}_k(p)}{\bar{F}_k(p)}\right)$ is a decreasing function on interval $p \in [v_k^*, \bar{u}_k]$.
- (2) $\bar{\Psi}_k(p) = \bar{\Psi}_k = \ln\left(1 + \frac{v_k^* \bar{q}_k}{1 - \bar{q}_k}\right)$ when $p \in (0, v_k^*)$.
- (3) $\bar{\Psi}_k(p) = 0$ when $p \in (\bar{u}_k, \infty)$.

The next Main Lemma 3 gives two important observations.

MAIN LEMMA 3 (VIRTUAL VALUE). For any feasible distribution F_k of Program (P1) and any potential-decrease $\Delta_k \in [0, \Psi_k]$, under the construction $\bar{F}_k \leftarrow \text{DIMINISH}(F_k, \Delta_k)$:

- (1) The new distribution \bar{F}_k is regular, i.e., $\bar{F}_k \in \text{REG}$.
- (2) The virtual value function $\bar{\varphi}_k(p) \geq \varphi_k(p)$ for all $p \in \mathbb{R}_{\geq 0}$.

3.5.3 Feasibility Analysis: Constraints (C2) and (C3). We next justify the feasibility to constraints (C2) and (C3) under the construction DIMINISH. For ease of reference, the both constraints are reformulated in the below.

$$r_i(q) = \frac{r_i(q_i)}{1 - q_i} \cdot (1 - q) \text{ for all } q \in [q_i, 1]; \quad (\text{C2})$$

$$v_i^* > 1, \varphi_i(v_i^{*+}) \geq 1, 0 < q_i < r_i(q_i) \leq 1, r_i(0) = 0. \quad (\text{C3})$$

[Constraint (C2)]. As mentioned in Section 3.5.2, the new distribution \bar{F}_i satisfies Fact 6.2: $\ln\left(1 + \frac{v_i^* \bar{q}_i}{1 - \bar{q}_i}\right) = \bar{\Psi}_i(p) = \ln\left(1 + p \cdot \frac{1 - \bar{F}_i(p)}{\bar{F}_i(p)}\right)$ for all $p \in [0, v_i^*]$. In terms of the revenue-quantile curve (see Section 2.1), we have $\ln\left(1 + \frac{\bar{r}_i(q)}{1 - q}\right) = \ln\left(1 + \frac{\bar{r}_i(\bar{q}_i)}{1 - \bar{q}_i}\right)$ for all $q \in [\bar{q}_i, 1]$. Then, elementary calculations indicate that constraint (C2) holds for the new distribution \bar{F}_k .

[Constraint (C3)]. We verify all inequalities involved in constraint (C3) one by one.

- (1) As mentioned in Section 3.5.1, the monopoly price v_i^* is invariant under the construction DIMINISH. Thus, the first inequality $v_i^* > 1$ trivially holds.
- (2) It follows from Main Lemma 3.2 that the monopoly virtual value $\bar{\varphi}_i(v_i^{**}) \geq \varphi_i(v_i^{**}) \stackrel{(C3)}{\geq} 1$.
- (3) As mentioned in Section 3.5.1, the monopoly quantile $\bar{q}_i \leq q_i$. When $\bar{q}_i > 0$, the revenue-quantile curve $\bar{r}_i(\bar{q}_i) = v_i^* \bar{q}_i > \bar{q}_i$ and $\bar{r}_i(\bar{q}_i) = v_i^* \bar{q}_i \leq v_i^* q_i = r_i(q_i) \leq 1$. When $\bar{q}_i = 0$, the new CDF $\bar{F}_i(p) = 1$ for all $p \in \mathbb{R}_{\geq 0}$, i.e., the new distribution \bar{F}_i is negligible.
- (4) As mentioned in Section 3.5.1, $\bar{r}_i(q) \leq r_i(q)$ for all $q \in [0, 1]$, hence the last inequality.

For simplicity, we never explicitly mention constraints (C2) and (C3) hereafter.

3.6 Construction of a New Instance $\bar{F} \cup \text{CONT}(\bar{y})$

To clarify the construction of a new instance (which is built on the construction DIMINISH), we shall generalize the notions about *potential* from a distribution to an instance:

DEFINITION 4 (POTENTIAL OF AN INSTANCE). For any regular instance $F = \{F_i\}_{i=1}^n \in \text{REG}^n$:

- Define the potential function $\Psi(p) \stackrel{\text{def}}{=} \sum_{i=1}^n \ln(1 + p \cdot \frac{1-F_i(p)}{F_i(p)})$ on interval $p \in \mathbb{R}_{\geq 0}$.
- Define the gross potential $\Psi \stackrel{\text{def}}{=} \sum_{i=1}^n \ln(1 + \frac{v_i q_i}{1-q_i})$.

As an implication of Fact 6, the next Fact 7 is tailored to the feasible instances of Program (P1).

FACT 7 (POTENTIAL OF AN INSTANCE). For any feasible instance $F = \{F_i\}_{i=1}^n$ of Program (P1):

- (1) The potential function $\Psi(p)$ is a decreasing function on interval $p \in \mathbb{R}_{\geq 0}$.
- (2) $\Psi(p) = \Psi$ when $0 \leq p \leq v^* \stackrel{\text{def}}{=} \min_{i \in [n]} \{v_i^*\}$.
- (3) $\Psi(p) = 0$ when $\infty > p > U \stackrel{\text{def}}{=} \max_{i \in [n]} \{u_i\}$.

3.6.1 Construction of $\{\bar{F}_i\}_{i=1}^n$: DIMINISH. Given any feasible instance $F = \{F_i\}_{i=1}^n$ of Program (P1), the construction of a new instance $\bar{F} = \{\bar{F}_i\}_{i=1}^n$ is controlled by a potential-decrease of $\Delta \in [0, \Psi]$, where $\Psi = \sum_{i=1}^n \ln(1 + \frac{v_i q_i}{1-q_i})$ is the gross potential of the given instance F . More concretely:

- We partition the potential-decrease of Δ into sub-potential-decreases $\{\Delta_i\}_{i=1}^n$. Recall Fact 7.1 that the potential function $\Psi(p) = \sum_{i=1}^n \ln(1 + \frac{v_i q_i}{1-q_i})$ is a decreasing function. Parameterized by $\bar{U} \stackrel{\text{def}}{=} \Psi^{-1}(\Delta) \in [v^*, U]$, we simply choose $\Delta_i \leftarrow \Psi_i(\bar{U})$ for each $i \in [n]$. Hence, (i) each sub-potential-decrease Δ_i is capped to the gross potential $\Psi_i = \ln(1 + \frac{v_i q_i}{1-q_i})$ of that distribution F_i ; and (ii) the potential-decrease of Δ entirely gets allocated.
- With the sub-potential-decreases $\{\Delta_i\}_{i=1}^n$, we obtain the new instance \bar{F} via (for all $k \in [n]$) the sub-constructions

$\bar{F}_k \leftarrow \text{DIMINISH}(F_k, \Delta_k)$. The parameter \bar{U} is exactly the support-supremum of the new instance \bar{F} .

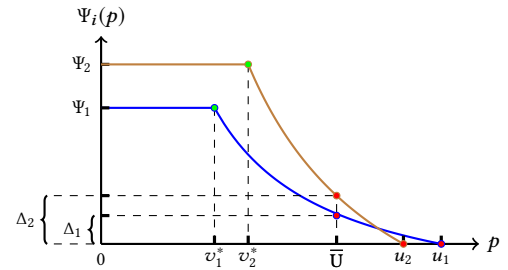
Formally, the partition scheme $\{\Delta_i\}_{i=1}^n$ and the construction are implemented as Algorithm 1 (which is also named after “DIMINISH” for simplicity) in the below. To make things mimic, we offer the subsequent Figure 6 for demonstration.

Algorithmus 1 DIMINISH(F, Δ)

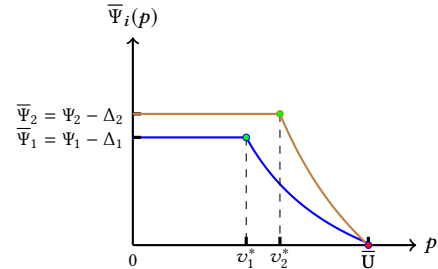
Input: instance $F = \{F_i\}_{i=1}^n$; potential-decrease Δ

Output: instance $\bar{F} = \{\bar{F}_i\}_{i=1}^n$

- 1: Define the new support-supremum $\bar{U} \leftarrow \Psi^{-1}(\Delta)$
 - 2: **for all** $k = 1, 2, \dots, n$ **do**
 - 3: $\Delta_k \leftarrow \min \{\Psi_k(\bar{U}), \Delta\}$, then $\Delta \leftarrow (\Delta - \Delta_k)$
 - 4: $\bar{F}_k \leftarrow \text{DIMINISH}(F_k, \Delta_k)$
 - 5: **end for**
 - 6: **return** the new instance \bar{F}
-



(a) The potential functions $\Psi_1(p)$ and $\Psi_2(p)$



(b) The potential functions $\bar{\Psi}_1(p)$ and $\bar{\Psi}_2(p)$

Figure 6: Demonstration for the construction DIMINISH.

Recall Main Lemma 3.2: under the construction DIMINISH, the virtual value function $\bar{\varphi}_i$ (of each new distribution \bar{F}_i) maps any value $p \in \mathbb{R}_{\geq 0}$ to a higher virtual value than the virtual value function φ_i (of each given distribution F_i).

Conceivably, higher virtual values shall lead to a higher Myerson Auction revenue. However, the construction DIMINISH makes each new distribution \bar{F}_i stochastically dominated by the given distribution F_i , which incurs a revenue loss of Myerson Auction. To deal with this issue, the key idea is to add a *continuous instance* (see Definition 2) as compensation. In the subsequent Section 3.6.2, we will explain how to construct this continuous instance.

3.6.2 Construction of $\text{CONT}(\bar{\gamma})$: AUGMENT. Composing any instance $\mathbf{F} = \{F_i\}_{i=1}^n$ with a “fake” continuous instance $\text{CONT}(\infty)$ preserves the feasibility to Program (P1) and the Myerson Auction revenue. Consequently, any feasible instance $\mathbf{F} \cup \text{CONT}(\gamma)$ of Program (P1) w.l.o.g. contains a continuous component, for some parameter $\gamma \in [1, \infty]$. For any parameter $\Delta \in \mathbb{R}_{\geq 0}$, we construct the new continuous component $\text{CONT}(\bar{\gamma})$ from the given continuous component $\text{CONT}(\gamma)$ as follows:

$$\bar{\gamma} \stackrel{\text{def}}{=} \mathcal{R}^{-1}(\mathcal{R}(\gamma) + \Delta),$$

which is demonstrated in the following Figure 7. Actually, function \mathcal{R} is a strictly decreasing function on its support $p \in [1, \infty]$ and has a range of $\mathcal{R}[1, \infty] = \mathbb{R}_{\geq 0}$. Thus, the new continuous component $\text{CONT}(\bar{\gamma})$ is always well defined. We call the construction “AUGMENT the continuous component $\text{CONT}(\gamma)$ by a factor of Δ ” or simply $\text{CONT}(\bar{\gamma}) \leftarrow \text{AUGMENT}(\text{CONT}(\gamma), \Delta)$.

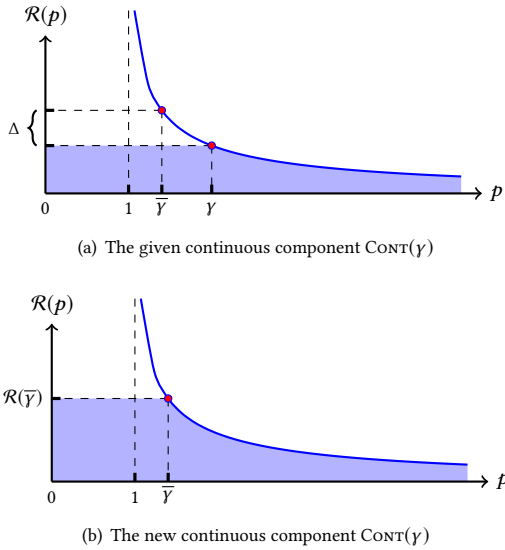


Figure 7: Demonstration for the construction AUGMENT.

3.6.3 Feasibility Analysis: Constraint (C1). For ease of reference, constraint (C1) is reformulated below, in terms of the potential function $\Psi(p) = \sum_{i=1}^n \ln(1 + p \cdot \frac{1-F_i(p)}{F_i(p)})$ and function $\mathcal{R}(p) = -p \cdot \ln(1-p^{-2})$. Notably, for any composition instance $\mathbf{F} \cup \text{CONT}(\gamma)$, the potential function $\Psi(p)$ only refers to the discrete component $\mathbf{F} = \{F_i\}_{i=1}^n$. Moreover, we slightly modify the LHS of constraint (C1) by taking into account the continuous component $\text{CONT}(\gamma)$.

$$\Psi(p) + \mathcal{R}(\max\{p, \gamma\}) \leq \mathcal{R}(p), \quad \forall p \in [1, \infty] \quad (\text{C1})$$

On the one hand, the construction $\bar{\mathbf{F}}_k \leftarrow \text{DIMINISH}(F_k, \Delta_k)$ slackens constraint (C1) by a factor of Δ . On the other hand, the construction $\text{CONT}(\bar{\gamma}) \leftarrow \text{AUGMENT}(\text{CONT}(\gamma), \Delta)$ tightens constraint (C1) by a factor of Δ . Thus, we can easily infer the feasibility.

3.6.4 Main Lemmas about DIMINISH and AUGMENT. We next offer two technical lemmas about the constructions DIMINISH and AUGMENT, which are the cornerstones of the proof of Theorem 1. The next Main Lemma 4 suggests that, under certain conditions, the constructions DIMINISH and AUGMENT as a whole lead to a higher Myerson Auction revenue.

MAIN LEMMA 4 (CONSTRUCTION). For any feasible instance $\mathbf{F} \cup \text{CONT}(\gamma)$ of Program (P1) and any constant $\varepsilon^* \in (0, 1/2)$, denote by $U \stackrel{\text{def}}{=} \max_{i \in [n]} \{u_i\}$ the support-supremum (of the discrete component $\mathbf{F} = \{F_i\}_{i=1}^n$) and by $V^* \stackrel{\text{def}}{=} \min_{i \in [n]} \{v_i^*\} > 1$ the minimum monopoly price. Suppose that the following holds:

- $\Psi(p) + \mathcal{R}(\max\{p, \gamma\}) \leq \mathcal{R}(p) - \varepsilon^*$ for all $p \in [1, U]$.
- $\mathcal{R}(\gamma) \geq \varepsilon^*$.

Then, for any potential-decrease⁴ $\Delta \leq \frac{1}{12} \cdot e^{-\frac{2}{V^*-1}} \cdot \varepsilon^{*3}$, consider the new instance $\bar{\mathbf{F}} \cup \text{CONT}(\bar{\gamma})$ obtained via $\bar{\mathbf{F}} \leftarrow \text{DIMINISH}(\mathbf{F}, \Delta)$ and $\text{CONT}(\bar{\gamma}) \leftarrow \text{AUGMENT}(\text{CONT}(\gamma), \Delta)$:

- (1) $\text{OPT}(\bar{\mathbf{F}} \cup \text{CONT}(\bar{\gamma})) \geq \text{OPT}(\mathbf{F} \cup \text{CONT}(\gamma))$.
- (2) $\bar{\Psi}(p) + \mathcal{R}(\max\{p, \bar{\gamma}\}) \leq \mathcal{R}(p) - \varepsilon^*$ for all $p \in (1, \bar{U}]$.
- (3) $\mathcal{R}(\bar{\gamma}) = \mathcal{R}(\gamma) + \Delta \geq \varepsilon^*$.

REMARK 1. Recall Section 3.6.1: to implement the construction $\bar{\mathbf{F}} \leftarrow \text{DIMINISH}(\mathbf{F}, \Delta)$, we first partition the potential-decrease Δ into sub-potential-decreases $\{\Delta_i\}_{i=1}^n$, and then obtain the new instance $\bar{\mathbf{F}}$ via the sub-constructions $\bar{\mathbf{F}}_k \leftarrow \text{DIMINISH}(F_k, \Delta_k)$ for each $k \in [n]$.

Naturally, the construction in Main Lemma 4 can be implemented as an n -round iterative algorithm: in each k -th round, we invoke the sub-constructions $\text{CONT}(\bar{\gamma}) \leftarrow \text{AUGMENT}(\text{CONT}(\gamma), \Delta_k)$ and $\bar{\mathbf{F}}_k \leftarrow \text{DIMINISH}(F_k, \Delta_k)$. It is not hard to see that each k -th round preserves **Points 2 to 4** (under minor modifications of the statements) in Main Lemma 4.

In fact, the Myerson Auction revenue increases in each k -th round. This task is much easier than proving **Point 1** directly: because only the k -th distribution and the continuous component change, we only need to reason about them. After settling this simplified task, we infer **Point 1** by induction.

Main Lemma 4 naturally leads to an induction-based proof of Theorem 1, since **Points 3** and **4** respectively preserve **Conditions (a)** and **(b)**. More concretely, once modifying the given instance $\mathbf{F} \cup \text{CONT}(\infty)$ so as to achieve the both conditions, we can construct the targeted continuous instance $\text{CONT}(\gamma^*)$ via an iterative algorithm:

- (1) Modify the given instance $\mathbf{F} \cup \text{CONT}(\infty)$ to achieve **Conditions (a)** and **(b)**.
- (2) Select a suitable step-size $\Delta > 0$ to make Main Lemma 4 applicable.
- (3) Invoke the constructions in Main Lemma 4 repeatedly.

⁴Also, Δ is capped to $\Psi = \sum_{i=1}^n \ln(1 + \frac{v_i^* q_i}{1-q_i})$, i.e., the gross potential of the discrete component \mathbf{F} .

By Main Lemma 4.1 and induction, the Myerson Auction revenue increases during **Step 3** of the iterative algorithm. Thus, to prove Theorem 1, the only issue left is whether the modification in **Step 1** incurs a large revenue loss of Myerson Auction. With a small enough constant ε^* in Main Lemma 4, we actually find a modification scheme so that the revenue loss is arbitrarily small, which is sufficient to establish Theorem 1:

- (1) **DIMINISH** the given discrete component F by a factor of $2 \cdot \varepsilon^*$.
- (2) **AUGMENT** the “fake” continuous component $\text{CONT}(\infty)$ by a factor of ε^* .

MAIN LEMMA 5 (MODIFICATION). *For any feasible instance $F = \{F_i\}_{i=1}^n$ of Program (P1) and any constant⁵ $\varepsilon^* \in (0, 1/2)$, consider the instance $\bar{F} \cup \text{CONT}(\bar{\gamma})$ obtained via the constructions $\bar{\gamma} \leftarrow \mathcal{R}^{-1}(\varepsilon^*)$*

and $\bar{F} \leftarrow \text{DIMINISH}(F, 2 \cdot \varepsilon^*)$:

- (1) $\text{OPT}(\bar{F} \cup \text{CONT}(\bar{\gamma})) \geq \text{OPT}(F \cup \text{CONT}(\infty)) - 8 \cdot \varepsilon^*$.
- (2) $\bar{\Psi}(p) + \mathcal{R}(\max\{p, \bar{\gamma}\}) \leq \mathcal{R}(p) - \varepsilon^*$ for all $p \in (1, \bar{U}]$.
- (3) $\mathcal{R}(\bar{\gamma}) = \varepsilon^*$.

3.7 Construction of the Targeted Continuous Instance $\text{CONT}(\gamma^*)$

Based on the former discussions, we now construct the targeted continuous instance $\text{CONT}(\gamma^*)$ from the given instance $F = \{F_i\}_{i=1}^n$. The whole construction is implemented as Algorithm 2 (named as MAIN), which is composed of two subroutines:

- (1) **PREPROCESS**, i.e., the modification of the given instance based on Main Lemma 5.
- (2) **FOR-LOOP**, i.e., the repeat invocation of the construction from Main Lemma 4.

We adopt the following notations in MAIN. Particularly, any quantity marked with an asterisk $*$ is invariant (e.g., each monopoly price $v_i^* > 1$, as mentioned in Section 3.5.1).

- $\Psi^* \stackrel{\text{def}}{=} \sum_{i=1}^n \ln(1 + \frac{v_i^* q_i}{1 - q_i})$ is the gross potential of the input instance.
- $\varepsilon^* < \min\{1/2, \Psi^*/2\}$ is an arbitrarily small constant.
- $V^* \stackrel{\text{def}}{=} \min_{i \in [n]} \{v_i^*\} > 1$ is the minimum monopoly price.
- $T^* \stackrel{\text{def}}{=} \lceil \frac{12}{\varepsilon^{*3}} \cdot \Psi^* \cdot e^{2/(V^*-1)} \rceil$ is the time horizon of **FOR-LOOP**.
- $\Delta^* \stackrel{\text{def}}{=} (\Psi^* - 2 \cdot \varepsilon^*)/T^*$ is the step size of **FOR-LOOP**.
- $\gamma^* \stackrel{\text{def}}{=} \mathcal{R}^{-1}(\Psi^* - 2 \cdot \varepsilon^*) \in [1, \infty]$ is well defined.

Finally, we can establish Theorem 1 by applying Algorithm 2 and Main Lemmas 4 and 5. During **PREPROCESS** (see **Step 1**), the gross potential of the discrete component decreases from Ψ^* to $(\Psi^* - 2 \cdot \varepsilon^*)$. Since the constant $\varepsilon^* < \min\{1/2, \Psi^*/2\}$, Main Lemma 5 is applicable, i.e., the revenue loss of Myerson Auction during **PREPROCESS** is at most $8 \cdot \varepsilon^*$.

Now, we can infer **Conditions (a)** and **(b)** in Main Lemma 4 respectively from **Points 3** and **4** in Main Lemma 5. By definition, $T^* \cdot \Delta^* = \Psi^* - 2 \cdot \varepsilon^*$, i.e., the remaining gross potential of $(\Psi^* - 2 \cdot \varepsilon^*)$

⁵Also, ε^* is capped to $\frac{1}{2} \cdot \sum_{i=1}^n \ln(1 + \frac{v_i^* q_i}{1 - q_i})$, i.e., half of the gross potential of the given discrete component F .

Algorithmus 2 MAIN($F^* \cup \text{CONT}(\infty), \varepsilon^*$)

Input: instance $F^* \cup \text{CONT}(\infty)$; constant $\varepsilon^* \in (0, 1/2)$

Output: continuous instance $\text{CONT}(\gamma^*)$

- 1: $\bar{F} \leftarrow \text{DIMINISH}(F, 2 \cdot \varepsilon^*)$ ▷ **PREPROCESS**
 - 2: $\text{CONT}(\bar{\gamma}) \leftarrow \text{AUGMENT}(\text{CONT}(\infty), \varepsilon^*)$
 - 3: Update: $F \cup \text{CONT}(\gamma) \leftarrow \bar{F} \cup \text{CONT}(\bar{\gamma})$
 - 4: **for all** $t = 1, 2, \dots, T^*$ **do** ▷ **FOR-LOOP**
 - 5: $\bar{F} \leftarrow \text{DIMINISH}(F, \Delta^*)$
 - 6: $\text{CONT}(\bar{\gamma}) \leftarrow \text{AUGMENT}(\text{CONT}(\gamma), \Delta^*)$
 - 7: Update: $F \cup \text{CONT}(\gamma) \leftarrow \bar{F} \cup \text{CONT}(\bar{\gamma})$
 - 8: **end for**
 - 9: **return** the continuous instance $\text{CONT}(\gamma^*) \leftarrow \text{CONT}(\gamma)$
-

is comprised of T^* units, with a potential of Δ^* each. In **FOR-LOOP** (see **Step 5**), each iteration $t \in [T^*]$ incurs a unit of potential decrease, i.e.,

$$\Delta^* = \frac{\Psi^* - 2 \cdot \varepsilon^*}{T^*} = \frac{\Psi^* - 2 \cdot \varepsilon^*}{\lceil \frac{12 \cdot \Psi^* \cdot e^{2/(V^*-1)}}{\varepsilon^{*3}} \rceil} < \frac{1}{12} \cdot e^{-2/(V^*-1)} \cdot \varepsilon^{*3}.$$

Thus, Main Lemma 4 is applicable to each of the T^* iterations, i.e., the Myerson Auction revenue never decreases during **FOR-LOOP**.

By Fact 5, this targeted continuous instance $\text{CONT}(\gamma^*)$ gives a smaller Myerson Auction revenue than the special continuous instance $\text{CONT}(1)$, i.e., $\text{OPT}(\text{CONT}(\gamma^*)) \leq \text{OPT}(\text{CONT}(1)) = C^* \approx 2.62$. Put everything together:

$$\text{OPT}(F^*) - 8 \cdot \varepsilon^* \leq \text{OPT}(\text{CONT}(\gamma^*)) \leq C^* \approx 2.62.$$

As the constant $\varepsilon^* \in (0, \frac{1}{2})$ can be arbitrarily small, $\text{OPT}(F^*) \leq C^* \approx 2.62$ for any feasible instance F^* of Program (P1).

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