# The Query Complexity of Uniform Pricing

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## Abstract

Real-world pricing mechanisms are typically optimized using training data, a setting corresponding to the *pricing query complexity* problem in Mechanism Design. The previous work [11] studies the *single-distribution* case[1], with tight bounds of $\widetilde{\Theta}(\varepsilon^{-3})$ for a *general* distribution and $\widetilde{\Theta}(\varepsilon^{-2})$ for either a *regular* or *monotone-hazard-rate (MHR)* distribution, where $\varepsilon \in (0,1)$ denotes the (additive) revenue loss of a learned uniform price relative to the Bayesian-optimal uniform price.

This can be directly interpreted as "the query complexity of the *Uniform Pricing* mechanism, in the *single-distribution* case". Yet in the *multi-distribution* case, can the regularity and MHR conditions still lead to improvements over the tight bound $\widetilde{\Theta}(\varepsilon^{-3})$ for general distributions? We answer this question in the negative, by establishing a (near-)matching lower bound $\Omega(\varepsilon^{-3})$ for either *two regular distributions* or *three MHR distributions*.

We also address the *regret minimization* problem and, in comparison with the folklore upper bound $\widetilde{O}(T^{2/3})$ for general distributions (see, e.g., [13]), establish a (near-)matching lower bound $\Omega(T^{2/3})$ for either *two regular distributions* or *three MHR distributions*, via a black-box reduction. Again, this is in stark contrast to the tight bound $\widetilde{\Theta}(T^{1/2})$ for a single regular or MHR distribution.

## CCS Concepts

• **Theory of computation → Algorithmic mechanism design**; **Computational pricing and auctions**; *Query learning*; *Regret bounds.*

## Keywords

Mechanism Design, Revenue Maximization, Uniform Pricing, Query Complexity, Regret Minimization

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## 1 Introduction

Uniform Pricing serves as a foundational mechanism in both economic theory [1] and real-world markets [5]. It describes a scenario where a platform sets a uniform price for $n \geq 1$ buyers and the transaction succeeds when at least one buyer is willing to accept this uniform price. From e-commerce platforms pricing products to cloud providers setting subscription fees, Uniform Pricing's ubiquity and operational simplicity belie a crucial challenge, especially in repeated interactions:

> *How to learn — information-efficiently — an (approximately) revenue-optimal uniform price?*

The arguably "most reasonable" information revealed in (a trial of) Uniform Pricing is the success or failure of the transaction.[2] Accordingly, a platform can learn and optimize Uniform Pricing using *pricing queries* [10, 11, 13, 14], namely trials of this mechanism itself. The efficiency of this learning process is measured by two canonical metrics (see Sections 2 and B for their formal definitions):

- *Query Complexity:* How many trials does a (possibly adaptive) pricing strategy require to output an $\varepsilon \in (0,1)$ approximately revenue-optimal uniform price.
- *Minimax Regret:* Over $T \geq 1$ trials, compared with an (exactly) revenue-optimal uniform price, how much a (possibly adaptive) pricing strategy will lose in cumulative revenue.

### 1.1 Previous Works: The Single-Distribution Case

In the *single-distribution* case ($n = 1$), the works [11, 13] have obtained a clear set of conclusions:

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[1] The $\widetilde{\Theta}$ notation omits polylogarithmic factors.

[2] This is mainly because Uniform Pricing is a pricing mechanism, where buyers make take-it-or-leave-it decisions. In contrast, for a (truthful) auction mechanism, the arguably "most reasonable" information is samples of buyers' values. (For a survey of the "sample complexity of mechanism design" literature, the interested reader can refer to the early works [3, 6, 9] as well as the recent work [7] and the references therein.)

> *Good single-distribution structure (regularity/MHR) can greatly improve the learning efficiency.*

(i) For a single *general* distribution (without distributional structure), the query complexity scales as $\widetilde{\Theta}(\varepsilon^{-3})$, and the minimax regret scales as $\widetilde{\Theta}(T^{2/3})$.[1]
(ii) For a single *regular* distribution [12] or a single *monotone hazard rate (MHR)* distribution [2],[3] two standard distributional conditions in the literature, the query complexity improves to $\widetilde{\Theta}(\varepsilon^{-2})$, and the minimax regret improves to $\widetilde{\Theta}(T^{1/2})$. Basically, the regularity/MHR condition imposes "convexity-like properties"[4] and, thus, enables "binary-like search" for an (approximately) revenue-optimal price.

## 1.2 This Work: The Multi-Distribution Case

The above results can be seamlessly interpreted as the learning efficiency of Uniform Pricing, in the *single-distribution* case ($n = 1$). Instead, this work addresses the extension to the *multi-distribution* case ($n \geq 2$), i.e., a scenario with $n \geq 2$ (independent) distributions drawn from a common distribution class — general, regular, or MHR. Of particular interest is the following question:

> *Can good multi-distribution structure (regularity/ MHR) still improve the learning efficiency?*

The multi-distribution case ($n \geq 2$), apart from being a seamless generalization, is worth investigating in various additional aspects. Firstly, value distributions in real-world markets can rarely be of a single type. For example, ride-hailing platforms such as Uber and Lyft have multi-type user values; time-sensitive riders (e.g., for work) might prioritize speed over cost, while off-peak riders often prefer cheaper options. Also, online shopping platforms such as Amazon often tier prices (even on the same product) for different types of buyers; urgent buyers (needing next-day deliveries) must pay full price, bulk shoppers may get volume discounts, and deal-seekers can wait for flash sales.

In addition, from a technical perspective, the success or failure of (the transaction in) Uniform Pricing depends on the *highest value* $\max_{i \in [n]} v_i$ (across all distributions $v_i \sim F_i$ for $i \in [n]$) and the corresponding *first-order distribution* $F(p) = \prod_{i \in [n]} F_i(p)$. In this vein:
(i) For multiple *general* distributions, their first-order distribution $F(p)$ possesses no specific structure, so the tight query complexity $\widetilde{\Theta}(\varepsilon^{-3})$ and the tight minimax regret $\widetilde{\Theta}(T^{2/3})$ are the same as before.
(ii) For multiple *regular/MHR* distributions, their first-order distribution $F(p)$ may violate the regularity/ MHR condition but, to a certain extent, still possesses good structure.[5] Hence, it is important to determine whether such "moderate" distributional structures can still improve the learning efficiency.

## 1.3 Our Contributions

We answer the above questions on the learning efficiency of Uniform Pricing, establishing a sharp dichotomy between the single-distribution case ($n = 1$) and the multi-distribution case ($n \geq 2$):

> *Good multi-distribution structure (regularity/MHR)* ***cannot*** *improve the learning efficiency.*

Namely, in the multi-distribution case ($n \geq 2$), both the query complexity and the minimax regret for *regular/MHR* distributions (essentially) revert to those for *general* distributions. In more detail, our conceptual and technical contributions can be divided into the following three categories.

**1. Matching Lower Bounds for Multiple Well-Structured Distributions.** We establish the following hardness results for both metrics — query complexity and minimax regret — showing that good distributional structure provides (almost) no benefit in the multi-distribution case ($n \geq 2$):

- For $n \geq 2$ regular distributions, the regularity condition cannot help in learning Uniform Pricing:
  We prove a query complexity lower bound of $\Omega(\varepsilon^{-3})$ and a minimax regret lower bound of $\Omega(T^{2/3})$, matching the tight bounds $\widetilde{\Theta}(\varepsilon^{-3})$ and $\widetilde{\Theta}(T^{2/3})$ for general distributions.
- For $n \geq 3$ MHR distributions, even the stronger MHR condition cannot help learn Uniform Pricing:
  Again, we establish a matching query complexity lower bound of $\Omega(\varepsilon^{-3})$ and a matching minimax regret lower bound of $\Omega(T^{2/3})$.
- For $n = 2$ MHR distributions, we prove a query complexity lower bound of $\Omega(\varepsilon^{-5/2})$ and a minimax regret lower bound of $\Omega(T^{3/5})$, leaving small gaps relative to the tight bounds $\widetilde{\Theta}(\varepsilon^{-3})$ and $\widetilde{\Theta}(T^{2/3})$ for general/regular distributions.

Our hardness results contrast sharply with the single-distribution case ($n = 1$) [11, 13], where regularity/MHR significantly improves the learning efficiency (e.g., reducing the query complexity from $\widetilde{\Theta}(\varepsilon^{-3})$ to $\widetilde{\Theta}(\varepsilon^{-2})$). So, in more competitive scenarios, such as ride-hailing and online shopping with $n \geq 2$ distributions, a platform must be more careful about the design of its pricing strategies.

**2. Insights behind Lower-Bound Construction.** As noted, the learning of a revenue-optimal uniform price $p^{\mathrm{opt}}$ (say) relies on the underlying *first-order distribution* $F(p) = \prod_{i \in [n]} F_i(p)$ and the corresponding *revenue function* $R(p) = p \cdot (1 - F(p))$.

For a single regular/MHR distribution, its revenue function $R(p)$ turns out to exhibit "convexity-like properties" [13],[4] which then enables "binary-like search" of $p^{\mathrm{opt}}$ (or its good enough approximations). This accounts for the improvements to $\widetilde{\Theta}(\varepsilon^{-2})$ and $\widetilde{\Theta}(T^{1/2})$ (over the tight bounds $\widetilde{\Theta}(\varepsilon^{-3})$ and $\widetilde{\Theta}(T^{2/3})$ for general distributions).

For multiple regular/MHR distributions, to establish our hardness results (that match the tight bounds for general distributions), our lower-bound construction must break the above "convexity-like

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[3] A distribution $F$ satisfies the (relatively weaker) regularity condition [12] when its *virtual value function* $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is nondecreasing, and satisfies the (relatively stronger) MHR condition [2] when its *hazard rate function* $h(v) = \frac{f(v)}{1-F(v)}$ is nondecreasing; see Section 2 for more details.

[4] More precisely, for a *regular* distribution $F$, (in a parametric equation form) the *revenue-quantile curve* $(1-F(p), p \cdot (1-F(p)))$ is *concave*. And for a *MHR* distribution $F$, the *cumulative hazard rate function* $H(p) = -\ln(1 - F(p))$ is *convex*.

[5] For example, for multiple *regular/MHR* distributions, their first-order distribution $F(p)$ satisfies the *quasi-regular/quasi-MHR* condition — a natural relaxation/generalization of regularity/MHR introduced by the recent work [7].

A distribution $F$ satisfies the (relatively weaker) quasi-regularity condition [7] when its *"conditional expected"* virtual value function $\phi_{CE}(v) = \mathbb{E}[\phi(x) | x \leq v]$ is nondecreasing, and satisfies the (relatively stronger) quasi-MHR condition [7] when its *"normalized cumulative"* hazard rate function $h_{NC}(v) = \frac{1}{v}\int_0^v h(x)\mathrm{d}x$ is nondecreasing. So, the quasi-regularity (resp. quasi-MHR) condition relaxes the *pointwise monotonicity* of a virtual value function (resp. a hazard rate function) to *on-average monotonicity.*

properties". We achieve this by leveraging the competition across individual regular/MHR distributions. Namely, even if all distributions $F_i$ for $i \in [n]$ are regular/MHR, the revenue function $R(p)$ can have two features:

- *Global Flatness:* $R(p)$ varies by at most $O(\varepsilon)$, over a wide enough region $I$ promised to contain $p^{\mathrm{opt}}$.
- *Local Sharpness:* $R(p)$ jumps up by at least $\Omega(\varepsilon)$ on narrow enough sub-intervals of $I$, e.g., at $p^{\mathrm{opt}}$.

It turns out that the global flatness means "the verification of $p^{\mathrm{opt}}$'s revenue-optimality" is inefficient, and the local sharpness means "the search for (a narrow enough sub-interval that contains) $p^{\mathrm{opt}}$" is inefficient. Altogether, this lower-bound construction scheme is sufficient to establish our hardness results.

**3. A Unified Framework for Lower-Bound Analysis.** To obtain our hardness results, we have further developed a unified framework for lower-bound analysis, adapting it to the specific contexts considered. Specifically, this framework consists of four components:

- *Base Instance:* Construct a suitable base instance $F^*$, such that the revenue function $R^*$ is exactly flat (i.e., all uniform prices $p \in I$ are equally revenue-optimal) over a wide enough region $I$.
- *Hard Instances:* Each hard instance $F^k$ for $k = 1, 2, \dots, K$ (say $K = \Omega(\varepsilon^{-1})$) modifies the base instance $F^*$ on some sub-interval $I^k \subseteq I$, in such a way:
  (i) Each hard instance $F^k$ retains the same distributional structure (regularity/MHR) as $F^*$.
  (ii) $R^k(p) > R^*(p) + \varepsilon$ for some $p \in I^k$, some modified revenue can exceed the base by more than $\varepsilon$.
  (iii) These modification sub-intervals $I^k$ for $k = 1, 2, \dots, K$ are disjoint.
- *Reduction from Query Complexity to Instance-Identification:* To get an $\varepsilon$-approximately revenue-optimal uniform price (say) in each possibility $k \in [K]$, a pricing strategy must identify the actual modification sub-interval $I^k$ and the actual hard instance $F^k$. Furthermore, information-theoretic arguments show that a single hard instance $F^k$ requires $\Omega(\varepsilon^{-2})$ queries to identify. Thus, a combination of both arguments gives a query complexity lower bound of $K \cdot \Omega(\varepsilon^{-2}) = \Omega(\varepsilon^{-3})$ (say).
- *Reduction from Minimax Regret to Instance-Identification:* This reduction simply adapts the above one, from the query complexity problem to the minimax regret problem; see Section B for details.

We believe that our unified framework is general-purpose — Uniform Pricing as a canonical mechanism is a representative instantiation — and can find far more applications in future research.

**Organization.** For ease of presentation, we focus on the query complexity problem throughout Sections 2 to 4. In Section 2, we show preliminaries. In Section 3, we prove an $\Omega(\varepsilon^{-3})$ lower bound for $n \geq 2$ regular distributions. In Section 4, we prove an $\Omega(\varepsilon^{-3})$ lower bound for $n \geq 3$ MHR distributions. (In the full version, we also prove an $\Omega(\varepsilon^{-5/2})$ lower bound for $n = 2$ MHR distributions.) Finally, in Section B, we extend these query complexity lower bounds to the corresponding minimax regret lower bounds.

## 2 Preliminaries

For a positive integer $n \geq 1$, we denote $[n] := \{1, 2, \dots, n\}$.

### 2.1 Probability and Distribution

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a sequence of random variables $\boldsymbol{X} = (X_i)_{i \in [n]}$, let $\mathbb{P}_{\boldsymbol{X}}(A)$ denote the *pushforward measure* of $\mathbb{P}$ by these random variables $\boldsymbol{X}$, for a measurable set $A$:[6]

$$\mathbb{P}_{\boldsymbol{X}}(A) \;:=\; \mathbb{P}[\{\omega \in \Omega : \boldsymbol{X} \in A\}].$$

For a sub $\sigma$-algebra $\mathcal{F}' \subseteq \mathcal{F}$, we also denote by $\mathbb{P}_{\boldsymbol{X} \mid \mathcal{F}'}(A)$ the *conditional pushforward measure* given $\mathcal{F}'$:

$$\mathbb{P}_{\boldsymbol{X} \mid \mathcal{F}'}(A) \;:=\; \mathbb{P}[\{\omega \in \Omega : \boldsymbol{X} \in A\} \mid \mathcal{F}'].$$

Regarding a *single-dimensional distribution* $F$, without ambiguity, we abuse the notation $F$ also for its *cumulative distribution function (CDF)*. Throughout this paper, we consider left-continuous CDF's, namely $F(p) := \mathbb{P}_{v \sim F}[v < p]$ for $p \in [-\infty, +\infty]$; we prefer this shift from convention, since a buyer with a random value $v \sim F$ is willing to buy a price-$p$ item with probability $\mathbb{P}[v \geq p]$, rather than $\mathbb{P}[v > p]$.

Definitions 1 and 2 introduces two canonical distribution families, the family of *regular* distributions [12] and the family of *monotone-hazard-rate (MHR)* distributions [2] — a regular or MHR distribution $F$ always has a well-defined (generalized) *probability density function (PDF)* $f$.

**Definition 1** (Regular Distributions [12]). A distribution $F$ satisfies the *regularity* condition when its virtual value function $\phi(x) := x - \frac{1-F(x)}{f(x)}$ is *nondecreasing* over its support.

**Definition 2** (MHR Distributions [2]). A distribution $F$ satisfies the *monotone-hazard-rate (MHR)* condition when its hazard rate function $h(x) := \frac{f(x)}{1-F(x)}$ is *nondecreasing* over its support.

There are other alternative/equivalent definitions of these two conditions; for details, the interested reader can reference the textbook [8] and the recent work [7]. By definitions, a MHR distribution must be a regular distribution, but the converse is incorrect in general; distributions like $F(x) = \max\left(\frac{x-1}{x}, 0\right)$ for $x \in [0, +\infty]$ are regular but non-MHR.

In addition, Claim 3 presents *Jensen's Inequality* in the context of probability theory.

**Claim 3** (Jensen's Inequality [4, Theorem 2.6.2]). *If $X$ is a random variable and $g$ is a convex function, then $g(\mathbb{E}[X]) \leq \mathbb{E}[g(X)]$.*

### 2.2 Uniform Pricing

In single-item mechanism design, a seller aims to sell an indivisible item to $n \geq 1$ buyers with *independent* value distributions $\boldsymbol{F} = \otimes_{i=1}^{n} F_i$. Specifically, the Uniform Pricing mechanism posts a uniform

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[6] For readers unfamiliar with this notion, when $\boldsymbol{X}$ consists of a single random variable $X_1$, we can informally interpret $\mathbb{P}_{\boldsymbol{X}}$ as the marginal probability of $X_1$ when $X_1$ is discrete, or the marginal density of $X_1$ when $X_1$ is absolutely continuous with respect to the Lebesgue measure. We adopt this more general measure-theoretic notion since the random variables $\boldsymbol{X} = (X_i)_{i \in [n]}$ might be neither discrete nor absolutely continuous.

price $p \geq 0$ on the item and sells it to any buyer (such as the first coming one) willing to pay this price; this results in the *first-order value distribution* $F$ and the *revenue function* $R$.

$$\begin{aligned} F(p) \ &:= \ \mathbb{P}_{\boldsymbol{v} \sim \boldsymbol{F}}\big[(\max_{i \in [n]} v_i) < p\big] \ = \ \textstyle\prod_{i=1}^{n} F_i(p), && \forall p \geq 0, \\ R(p) \ &:= \ p \cdot \mathbb{P}_{\boldsymbol{v} \sim \boldsymbol{F}}\big[(\max_{i \in [n]} v_i) \geq p\big] \ = \ p \cdot \big(1 - F(p)\big), && \forall p \geq 0. \end{aligned}$$

In the bulk of this paper, we will study (a generalized version of) the *pricing query complexity* problem; to make the problem interesting, we follow the previous works [10, 11, 14] and consider $[0, 1]$-supported value distributions. (Thus, the *optimal uniform price* $p^{\text{opt}} = p^{\text{opt}}(\boldsymbol{F}) := \operatorname{argmax}_{p \in [0,1]} R(p)$ is well-defined and lies in the support $[0, 1]$.) A pricing algorithm $\mathcal{A}$ works as follows:

- At the beginning, $\mathcal{A}$ has no information of the value distributions $\boldsymbol{F}$ (except for their independence and $[0, 1]$ support).
- $\mathcal{A}$ acquires information of the value distributions $\boldsymbol{F}$ using *pricing queries*.[7] Each time $t = 1, 2, \dots$, $\mathcal{A}$ posts a *query price* $p^t$ and acquires *binary feedback* $z^t = z^t(p^t) := \mathbb{1}[(\max_{i \in [n]} v_i^t) \geq p^t] \in \{0, 1\}$ based on an independent draw $\boldsymbol{v}^t \sim \boldsymbol{F}$; this gives $\mathbb{P}[z^t = 0] = F(p^t)$ and $\mathbb{P}[z^t = 1] = 1 - F(p^t)$, i.e., an independent trial of the Uniform Pricing mechanism — whether the sale using a uniform price $p^t$ succeeds or not.
- At the termination, $\mathcal{A}$ needs to output a price $p^{\mathcal{A}}$.

The pricing query complexity problem asks, over the randomness of both the value distributions $\boldsymbol{F}$ and the pricing algorithm $\mathcal{A}$ itself, *how many pricing queries are sufficient and necessary to succeed in outputting a "good enough" price* $p^{\mathcal{A}}$: given $\varepsilon \in (0, 1)$,

$$\mathbb{P}_{\boldsymbol{F}, \mathcal{A}}\big[R(p^{\mathcal{A}}) \geq R(p^{\text{opt}}) - \varepsilon\big] \ \geq \ \tfrac{2}{3}.$$

Here, the $\frac{2}{3}$ success probability is a standard convention in designing probabilistic algorithms (and can be replaced by any other constant strictly larger than $\frac{1}{2}$).

Later in Section B, we will study another related problem, the *regret minimization* problem.

## 2.3 Information Theory

Let $\mathbb{P}^*$ and $\mathbb{P}$ be two probability measures on the same measurable space $(\Omega, \mathcal{F})$. When $\mathbb{P}^*$ is absolutely continuous with respect to $\mathbb{P}$, their *Kullback-Leibler (KL) divergence* [4, Chapter 2.3] is given by

$$\mathsf{KL}(\mathbb{P}^*, \mathbb{P}) \ := \ \mathbb{E}_{\mathbb{P}^*}\Big[\ln\Big(\tfrac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\Big)\Big],$$

where $\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}$ is the Radon-Nikodym derivative. Without ambiguity, we abuse the notation $\mathsf{KL}(p, q)$ to denote the KL divergence between two Bernoulli distributions with parameters $p, q \in [0, 1]$:

$$\mathsf{KL}(p, q) \ := \ p \ln\Big(\tfrac{p}{q}\Big) + (1 - p) \ln\Big(\tfrac{1-p}{1-q}\Big).$$

Claims 4 to 6 will be useful in our later proofs.

**Claim 4** (Convexity of $\mathsf{KL}(p, q)$). *The function* $\mathsf{KL}(p, q)$ *is convex on* $(p, q) \in [0, 1]^2$.

Proof. By elementary algebra, the function $\mathsf{KL}(p, q)$'s Hessian matrix $\boldsymbol{H}_{\mathsf{KL}}$ is given by

$$\boldsymbol{H}_{\mathsf{KL}} \ := \ \begin{bmatrix} \frac{\partial^2 \mathsf{KL}}{\partial p^2} & \frac{\partial^2 \mathsf{KL}}{\partial p \partial q} \\ \frac{\partial^2 \mathsf{KL}}{\partial q \partial p} & \frac{\partial^2 \mathsf{KL}}{\partial q^2} \end{bmatrix} \ = \ \begin{bmatrix} \frac{1}{p} + \frac{1}{1-p} & -\frac{1}{q} - \frac{1}{1-q} \\ -\frac{1}{q} - \frac{1}{1-q} & \frac{p}{q^2} + \frac{1-p}{(1-q)^2} \end{bmatrix}.$$

Since $\boldsymbol{H}_{\mathsf{KL}}$ has *nonnegative* diagonal elements as well as a *nonnegative* determinant

$$\begin{aligned} |\boldsymbol{H}_{\mathsf{KL}}| \ &= \ \big(\tfrac{1}{p} + \tfrac{1}{1-p}\big) \cdot \big(\tfrac{p}{q^2} + \tfrac{1-p}{(1-q)^2}\big) - \big(\tfrac{1}{q} + \tfrac{1}{1-q}\big)^2 \\ &= \ \tfrac{1-p}{p \cdot (1-q)^2} + \tfrac{p}{(1-p) \cdot q^2} - \tfrac{2}{q \cdot (1-q)} \ \geq \ 0, \end{aligned}$$

the function $\mathsf{KL}(p, q)$ is *convex* on $(p, q) \in [0, 1]^2$. This finishes the proof of Claim 4. □

**Claim 5** (Upper Bounds of $\mathsf{KL}(p, q)$). $\mathsf{KL}(p, q) \leq 3 \cdot (p - q)^2$, *for* $p \in [\frac{1}{7}, \frac{6}{7}]$ *and* $q \in [p - \frac{1}{12}, p]$.

Proof. For notational brevity, let $\delta := p - q \in [0, \frac{1}{12}]$; note that $\frac{\delta}{p}, \frac{\delta}{1-p} \in [0, \frac{7}{12}]$. We deduce that

$$\begin{aligned} \mathsf{KL}(p, q) \ &= \ -p \ln\Big(1 - \tfrac{\delta}{p}\Big) - (1 - p) \ln\Big(1 + \tfrac{\delta}{1-p}\Big) \\ &\leq \ p \cdot \Big(\tfrac{\delta}{p} + \tfrac{\delta^2}{p^2}\Big) - (1 - p) \cdot \Big(\tfrac{\delta}{1-p} - \tfrac{1}{2} \cdot \tfrac{\delta^2}{(1-p)^2}\Big) \\ &= \ \tfrac{\delta^2}{p} + \tfrac{\delta^2}{2 \cdot (1-p)} \ \leq \ 3\delta^2. \end{aligned}$$

Here the second step uses $-\ln(1 - x) \leq x + x^2$ and $\ln(1 + x) \geq x - \frac{1}{2}x^2$, for $x \in [0, \frac{7}{12}]$. And the last step uses $\frac{1}{p} + \frac{1}{2 \cdot (1-p)} \leq \frac{3}{2} + \sqrt{2} \approx 2.9142$. This finishes the proof of Claim 5. □

**Claim 6** (Pricing Algorithms). *Consider two instances* $\boldsymbol{F}^*$ *and* $\boldsymbol{F}$ *that their first-order CDF's are identical,* $F(p) = F^*(p)$ *for* $p \notin P$, *everywhere except for a measurable subset* $P \subseteq [0, 1]$.

*Let* $\mathbb{P}^*$ *and* $\mathbb{P}$ *be the probability measures, on the same measurable space* $(\Omega, \mathcal{F})$, *by running a pricing algorithm* $\mathcal{A}$ *respectively on* $\boldsymbol{F}^*$ *and* $\boldsymbol{F}$. *Then, for any random event* $\mathcal{E} \in \mathcal{F}$,

$$\mathsf{KL}\big(\mathbb{P}^*[\mathcal{E}], \ \mathbb{P}[\mathcal{E}]\big) \ \leq \ \mathbb{E}_{\mathbb{P}^*}[T^P] \cdot \big(\max_{p \in P} \mathsf{KL}(F^*(p), \ F(p))\big).$$

*where* $T^P := |\{t \in [T] \,:\, p^t \in P\}|$ *denotes how many pricing queries are made using query prices in* $P$.

Proof. Since the conclusion holds trivially when $\mathbb{E}\left[T^P\right] = \infty$, thus in the following proof we only consider the case $\mathbb{E}\left[T^P\right] < \infty$. By the *data-processing inequality* for KL divergence [4, Chapter 2.8],

$$\mathsf{KL}\big(\mathbb{P}^*[\mathcal{E}], \ \mathbb{P}[\mathcal{E}]\big) \ \leq \ \mathsf{KL}(\mathbb{P}^*, \ \mathbb{P}).$$

We denote by $\mathscr{H}^t := \{(p^\tau, z^\tau)\}_{\tau=1}^{t-1}$ the query prices posted and the binary feedback acquired before time $t$; let $\mathscr{H}^1 := \emptyset$ for notational consistency. We can deduce that

$$\begin{aligned} \mathsf{KL}(\mathbb{P}^*, \ \mathbb{P}) \ &= \ \mathbb{E}_{\mathbb{P}^*}\bigg[\textstyle\sum_{t \in [T]} \bigg(\ln\bigg(\frac{\mathrm{d}\mathbb{P}^*_{z^t | \mathscr{H}^t \cup \{p^t\}}}{\mathrm{d}\mathbb{P}_{z^t | \mathscr{H}^t \cup \{p^t\}}}\bigg) + \ln\bigg(\frac{\mathrm{d}\mathbb{P}^*_{p^t | \mathscr{H}^t}}{\mathrm{d}\mathbb{P}_{p^t | \mathscr{H}^t}}\bigg)\bigg)\bigg] \\ &= \ \mathbb{E}_{\mathbb{P}^*}\bigg[\textstyle\sum_{t \in [T]} \ln\bigg(\frac{\mathrm{d}\mathbb{P}^*_{z^t | p^t}}{\mathrm{d}\mathbb{P}_{z^t | p^t}}\bigg)\bigg] \\ &= \ \mathbb{E}_{\mathbb{P}^*}\bigg[\textstyle\sum_{t \in [T] : p^t \in P} \ln\bigg(\frac{\mathrm{d}\mathbb{P}^*_{z^t | p^t}}{\mathrm{d}\mathbb{P}_{z^t | p^t}}\bigg)\bigg] \\ &= \ \mathbb{E}_{\mathbb{P}^*}\bigg[\textstyle\sum_{t \in [T] : p^t \in P} \mathbb{E}_{\mathbb{P}^*_{z^t | p^t}}\bigg[\ln\bigg(\frac{\mathrm{d}\mathbb{P}^*_{z^t | p^t}}{\mathrm{d}\mathbb{P}_{z^t | p^t}}\bigg)\bigg]\bigg] \end{aligned}$$

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[7] In this way, pricing algorithms come in two flavors, *adaptive* and *non-adaptive*: an adaptive one can determine a query price $p^t$ based on the information acquired thus far, while a non-adaptive one must determine all query prices $p^1, p^2, \dots$ in advance.

$$\le \ \mathbb{E}_{\mathbb{P}^*}\left[\textstyle\sum_{t\in[T]:\,p^t\in P}\big(\max_{p\in P}\mathsf{KL}(F^*(p),\,F(p))\big)\right]$$
$$= \ \mathbb{E}_{\mathbb{P}^*}[T^P]\cdot\big(\max_{p\in P}\mathsf{KL}(F^*(p),\,F(p))\big),$$

Here the third step holds since $F(p) = F^*(p)$ for $p \notin P$. And the last step uses *Wald's equation* [15].

□

## 3 $\Omega(\varepsilon^{-3})$ Lower Bound for Two Regular Distributions

In this section, we investigate the query complexity of Uniform Pricing in the setting with *regular* distributions. Specifically, we will establish Theorem 7.

**Theorem 7.** *For two (or more) regular distributions, the query complexity of Uniform Pricing is* $\Omega(\varepsilon^{-3})$.

*Remark.* This result is most relevant to the work [11], which shows:
(i) For *any number of general distributions*, the query complexity of Uniform Pricing is $\widetilde{\Theta}(\varepsilon^{-3})$.
(ii) For *a single regular distribution*, the query complexity of Uniform Pricing is $\widetilde{\Theta}(\varepsilon^{-2})$.
Consequently, Theorem 7 complements [11] by showing a more thorough picture: "a single regular distribution" is a quite singular case — even the *minimal generalization* to "two regular distributions" will increase the query complexity from $\widetilde{\Theta}(\varepsilon^{-2})$ to the general-case bound $\widetilde{\Theta}(\varepsilon^{-3})$.

In the remainder of this section, we will establish Theorem 7. Without loss of generality, we consider a sufficiently small $\varepsilon \in (0, \frac{1}{16})$ and a sufficiently large $K := \lfloor \frac{1}{2}\varepsilon^{-1} \rfloor \ge 8$ throughout. The entire proof takes two steps. Firstly, we present in Section 3.1 our lower-bound construction, including one base instance $\boldsymbol{F}^*$ and $K$ hard instances $\{\boldsymbol{F}^i\}_{i\in[K]}$. Afterward, we present in Section 3.2 our lower-bound analysis, including (i) a reduction from the original *pricing* problem to a new *instance-identification* problem and (ii) a proof of a matching lower bound $\Omega(\varepsilon^{-3})$ for the new problem. (Notably, later in Section 4, we will extend these lower-bound construction and analysis to the setting with MHR distributions.)

### 3.1 Lower Bound Construction

Consider two parameters $t \in [\frac{1}{2} + \varepsilon, 1]$ and $s = s(t) := t - \varepsilon$. We define two parametric CDF's $F_1^{*t}$ and $F_2^{*t}$ (cf. Figure 1(a)):

$$F_1^{*t}(x) \ := \ \begin{cases} \frac{x}{x+\frac{t}{3t-1}}, & x \in [0, t] \\ \frac{x-\frac{1}{3}}{x}, & x \in (t, 1] \end{cases},$$

$$F_2^{*t}(x) \ := \ \begin{cases} 0, & x \in [0, \frac{1}{3}] \\ \frac{x-\frac{1}{3}}{x}\cdot\frac{x+\frac{t}{3t-1}}{x}, & x \in (\frac{1}{3}, t] \\ 1, & x \in (t, 1] \end{cases}.$$

(We will verify the regularity of $F_1^{*t}$ and $F_2^{*t}$ later in Lemma 8.) In regard to Uniform Pricing, the first-order CDF $F^{*t}(x) := F_1^{*t}(x)\cdot F_2^{*t}(x)$ and the revenue function $R^{*t}(x) := x\cdot(1 - F^{*t}(x))$ are given as follows (cf. Figure 1(c)):

$$F^{*t}(x) \ = \ \begin{cases} 0, & x \in [0, \frac{1}{3}] \\ \frac{x-\frac{1}{3}}{x}, & x \in (\frac{1}{3}, 1] \end{cases}, \qquad R^{*t}(x) \ = \ \begin{cases} x, & x \in [0, \frac{1}{3}] \\ \frac{1}{3}, & x \in (\frac{1}{3}, 1] \end{cases}.$$

These formulae $F^{*t}$ and $R^{*t}$ are independent of $t \in [\frac{1}{2} + \varepsilon, 1]$; accordingly, we can define our base instance $\boldsymbol{F}^* := F_1^{*t} \otimes F_2^{*t}$ using any specific $t \in [\frac{1}{2} + \varepsilon, 1]$.

To establish the desired lower bound $\Omega(\varepsilon^{-3})$, we modify the second parametric CDF $F_2^{*t}$ into another parametric CDF $F_2^t$ on the interval $(s, t] = (t - \varepsilon, t]$ (cf. Figure 1(b)), as follows. (In contrast, the first parametric CDF $F_1^{*t}$ keeps the same.)

$$F_2^t(x) \ := \ \begin{cases} 1 - \frac{(1-F_2^{*t}(s))^2}{(x-s)\cdot f_2^{*t}(s)+(1-F_2^{*t}(s))} \\ \qquad = \ 1 - \frac{(1-F_2^{*t}(s))^2/f_2^{*t}(s)}{x-\phi_2^{*t}(s)}, & x \in (s, t] \\ F_2^{*t}(x), & x \notin (s, t] \end{cases}.$$

(I.e., only pricing queries within the modification interval $(s, t]$ can help identify $F_2^t$.) Then, the modified first-order CDF $F^t(x) := F_1^{*t}(x)\cdot F_2^t(x)$ and the modified revenue function $R^t(x) := x\cdot(1 - F^t(x))$ follow accordingly (cf. Figure 1(d)).

*Remark.* The modified parametric CDF $F_2^t$ is defined such that *"over the interval $x \in (s, t]$, the corresponding virtual value function $\phi_2^t(x)$ is constant $= \phi_2^{*t}(s)$"*; below, the proof of Lemma 8 will show this explicitly. This induces an ordinary differential equation (ODE); solving it under the boundary condition $\lim_{x\to s^+} F_2^t(x) = F_2^{*t}(s)$ (so as to preserve the continuity at $x = s$) gives the above defining formula.

Lemma 8 verifies the regularity of all considered CDF's.

**Lemma 8** (Regularity)**.** *Given any $t \in [\frac{1}{2} + \varepsilon, 1]$, all of $F_1^{*t}$, $F_2^{*t}$, and $F_2^t$ are well-defined regular CDF's.*

Proof. The first function $F_1^{*t}$ is *continuous* at $x = t$ (as $\lim_{x\to t^+} F_1^{*t}(x) = \frac{t-\frac{1}{3}}{t} = F_1^{*t}(t)$), has a *nonnegative* derivative function $f_1^{*t}$ (as follows), takes values between $0 = F_1^{*t}(0) \le F_1^{*t}(1) = \frac{2}{3}$, and has a *nondecreasing* virtual value function $\phi_1^{*t}$ (as follows). Given these, $F_1^{*t}$ is a well-defined regular CDF.

$$f_1^{*t}(x) \ = \ F_1^{*t\prime}(x) \ = \ \begin{cases} \frac{\frac{t}{3t-1}}{(x+\frac{t}{3t-1})^2}, & x \in [0, t] \\ \frac{1}{3x^2}, & x \in (t, 1] \end{cases},$$

$$\phi_1^{*t}(x) \ = \ x - \frac{1-F_1^{*t}(x)}{f_1^{*t}(x)} \ = \ \begin{cases} -\frac{t}{3t-1}, & x \in [0, t] \\ 0, & x \in (t, 1] \end{cases}.$$

The second function $F_2^{*t}$ is $(\frac{1}{3}, t]$-supported, is *continuous* at both endpoints $\lim_{x\to(\frac{1}{3})^+} F_2^{*t}(x) = 0 = F_2^{*t}(\frac{1}{3})$ and $\lim_{x\to t^+} F_2^{*t}(x) = 1 = F_2^{*t}(t)$, has a *nonnegative* derivative function $f_2^{*t}$ (as follows), takes values between $0 = F_2^{*t}(\frac{1}{3}) \le F_2^{*t}(t) = 1$, and has a *nondecreasing* virtual value function $\phi_2^{*t}$ (as follows). Given these, $F_2^{*t}$ is a well-defined regular CDF.

$$f_2^{*t}(x) \ = \ F_2^{*t\prime}(x) \ = \ \frac{2t-x}{3x^3\cdot(3t-1)}, \qquad x \in (\tfrac{1}{3}, t],$$
$$\phi_2^{*t}(x) \ = \ x - \frac{1-F_2^{*t}(x)}{f_2^{*t}(x)} \ = \ \frac{t\cdot x}{2t-x}, \qquad x \in (\tfrac{1}{3}, t].$$

The modified function $F_2^t$ is $(\frac{1}{3}, t]$-supported — with different defining formulae on the sub-intervals $(\frac{1}{3}, s]$ and $(s, t]$ — is *continuous* at the division point $\lim_{x\to s^+} F_2^t(x) = F_2^{*t}(s) = F_2^t(s)$, has a *nonnegative* derivative function $f_2^t$ (as follows), takes values between $0 = F_2^t(\frac{1}{3}) \le F_2^t(t) \le 1$, and has a *nondecreasing* virtual value

(a) "Base" (Second) CDF $F_2^{*t}$

(b) "Hard" (Second) CDF $F_2^t$

(c) "Base" Revenue Function $R^{*t}$

(d) "Hard" Revenue Function $R^t$

**Figure 1: Diagrams for the lower-bound construction in the "two regular distributions" setting.**

function $\phi_2^t$ (as follows). Given these, $F_2^t$ is a well-defined regular CDF.

$$f_2^t(x) = F_2^{t\prime}(x) = \begin{cases} \frac{(1-F_2^{*t}(s))^2/f_2^{*t}(s)}{(x-\phi_2^{*t}(s))^2}, & x \in (s,t] \\ f_2^{*t}(x), & x \notin (s,t] \end{cases},$$

$$\phi_2^t(x) = x - \frac{1-F_2^t(x)}{f_2^t(x)} = \begin{cases} x - \frac{\frac{(1-F_2^{*t}(s))^2/f_2^{*t}(s)}{x-\phi_2^{*t}(s)}}{\frac{(1-F_2^{*t}(s))^2/f_2^{*t}(s)}{(x-\phi_2^{*t}(s))^2}} = \phi_2^{*t}(s), & x \in (s,t] \\ \phi_2^{*t}(x), & x \notin (s,t] \end{cases}.$$

□

In addition, Lemmas 9 and 10 will be useful for our lower-bound analysis in Section 3.2.

**Lemma 9** (First-Order CDF's). *$0 \le F^{*t}(x) - F^t(x) \le \frac{4}{3}\varepsilon$ for $x \in (s,t]$, while $F^{*t}(x) = F^t(x)$ for $x \notin (s,t]$.*

Proof. Following the proof of Lemma 8, we have $\phi_2^{*t}(x) \ge \phi_2^{*t}(s) = \phi_2^t(x) \Leftrightarrow -\frac{f_2^{*t}(x)}{1-F_2^{*t}(x)} \le -\frac{f_2^t(x)}{1-F_2^t(x)} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\big(1 - F_2^{*t}(x)\big) \le \frac{\mathrm{d}}{\mathrm{d}x}\ln\big(1 - F_2^t(x)\big)$, for $x \in (s,t]$. Together with the boundary condition $F_2^{*t}(s) = F_2^t(s)$, we deduce that $F_2^{*t}(x) - F_2^t(x) \ge 0 \Rightarrow F^{*t}(x) - F^t(x) = F_1^{*t}(x) \cdot \big(F_2^{*t}(x) - F_2^t(x)\big) \ge 0$, for $x \in (s,t]$. Also, we have $f^{*t}(x) - f^t(x) \le f^{*t}(x) = \frac{1}{3x^2} \le \frac{4}{3}$, for $x \in (s,t] \subseteq [\frac{1}{2}, 1]$, where the first step uses the modified first-order PDF $f^t$'s nonnegativity. Together with the boundary condition $F^{*t}(s) = F^t(s)$, we deduce that $F^{*t}(x) - F^t(x) \le \frac{4}{3} \cdot (x - s) \le \frac{4}{3} \cdot (t - s) = \frac{4}{3}\varepsilon$, for $x \in (s,t]$.

The second part "$F^{*t}(x) = F^t(x)$ for $x \notin (s,t]$" is obvious. □

**Lemma 10** (Revenue Functions). *$R^t(t) \ge \frac{1}{3} + \frac{1}{18}\varepsilon$, while $R^t(x) \le \frac{1}{3}$ for $x \notin (s,t]$.*

Proof. Following the proof of Lemma 8, we have $F_1^{*t}(t) = \frac{t-\frac{1}{3}}{t}$, $F_2^{*t}(s) = \frac{s-\frac{1}{3}}{s} \cdot \frac{s+\frac{t}{3t-1}}{s} = 1 - \frac{\varepsilon}{3\cdot(t-\varepsilon)^2\cdot(3t-1)}$, and $f_2^{*t}(s) = \frac{2t-s}{3s^3\cdot(3t-1)} = \frac{t+\varepsilon}{3\cdot(t-\varepsilon)^3\cdot(3t-1)}$.

Given that $R^{*t}(t) = \frac{1}{3}$, we can deduce the first part $R^t(t) \ge \frac{1}{3} + \frac{1}{18}\varepsilon$, as follows:

$$\begin{aligned} R^t(t) - R^{*t}(t) &= t \cdot F_1^{*t}(t) \cdot (F_2^{*t}(t) - F_2^t(t)) \\ &= t \cdot F_1^{*t}(t) \cdot \frac{(1-F_2^{*t}(s))^2}{(t-s)\cdot f_2^{*t}(s)+(1-F_2^{*t}(s))} \\ &= t \cdot \frac{3t-1}{3t} \cdot \frac{\left(\frac{\varepsilon}{3\cdot(t-\varepsilon)^2\cdot(3t-1)}\right)^2}{\varepsilon\cdot\frac{t+\varepsilon}{3\cdot(t-\varepsilon)^3\cdot(3t-1)}+\frac{\varepsilon}{3\cdot(t-\varepsilon)^2\cdot(3t-1)}} \\ &= \frac{\varepsilon}{18t\cdot(t-\varepsilon)} \ \ge\ \frac{1}{18}\varepsilon. \end{aligned}$$

Here the first two steps substitutes the defining formulae of $R^t(t)$, $R^{*t}(t)$, and $F_2^t(t)$; note that $F_2^{*t}(t) = 1$. The third step substitutes the above formulae of $F_1^{*t}(t)$, $F_2^{*t}(s)$, and $f_2^{*t}(s)$. The fourth step rearranges the equation. And the last step holds since $\varepsilon \in (0, \frac{1}{16})$ and $t \in [\frac{1}{2} + \varepsilon, 1]$.

The second part "$R^t(x) = R^{*t}(x) \le \frac{1}{3}$ for $x \notin (s,t]$" is obvious. This finishes the proof of Lemma 10. □

To get the desired lower bound $\Omega(\varepsilon^{-3})$, we choose a sequence of *disjoint* modification intervals[8] $(s^i, t^i] := (\frac{1}{2} + i\varepsilon - \varepsilon, \frac{1}{2} + i\varepsilon]$ and, thus, obtain our hard instances $\boldsymbol{F}^i := F_1^{*t^i} \otimes F_2^{t^i}$ for $i \in [K]$.

## 3.2 Lower Bound Analysis

Consider a specific pricing algorithm $\mathcal{A}$ that performs well on all hard instances $\{\boldsymbol{F}^i\}_{i\in[K]}$: in any possibility $i \in [K]$, it always

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[8] Since $K = \lfloor \frac{1}{2}\varepsilon^{-1} \rfloor$, all parameters $\{t^i\}_{i\in[K]}$ belong to the interval $[\frac{1}{2} + \varepsilon, 1]$, satisfying the premises of Lemmas 8 and 10.

outputs a $\frac{1}{20}\varepsilon$-approximately optimal price $p^{\mathcal{A}}$ with probability $\geq \frac{2}{3}$:

$$R^i(p^{\mathcal{A}}) \;\geq\; \max\nolimits_{p\in[0,1]} R^i(p) - \tfrac{1}{20}\varepsilon.$$

(In contrast, no performance guarantee for the base instance $\boldsymbol{F}^*$ is needed.) Based on this, we can develop a "pricing-to-identification" reduction for all base/hard instances $\{\boldsymbol{F}^*\} \cup \{\boldsymbol{F}^i\}_{i\in[K]}$, i.e., another identification algorithm $\mathcal{B}^{\mathcal{A}}$ with *exactly the same number of pricing queries*:

- Run $\mathcal{A}$ on an unknown instance $\boldsymbol{F}$ (promised to be one of $\{\boldsymbol{F}^*\} \cup \{\boldsymbol{F}^i\}_{i\in[K]}$), getting a price $p^{\mathcal{A}}$.
- If $p^{\mathcal{A}} \in (s^i, t^i]$ for some $i \in [K]$, output $\boldsymbol{F}^i$;
- Otherwise, output $\boldsymbol{F}^*$.

Clearly, $\mathcal{B}^{\mathcal{A}}$ can identify all hard instances $\{\boldsymbol{F}^i\}_{i\in[K]}$:[9] in any possibility $i \in [K]$, it always successfully outputs $\boldsymbol{F}^i$ with probability $\geq \frac{2}{3}$. (Again, no performance guarantee for the base instance $\boldsymbol{F}^*$ is needed.)

In the rest of this section, we consider a specific identification algorithm $\mathcal{B}$; denote by $T$ the number of pricing queries it makes and, in particular, $T^i$ for $i \in [K]$ the number of pricing queries it makes within the index-$i$ modification interval $(s^i, t^i]$. Also, denote by $\mathbb{P}^*[\cdot]$ or $\mathbb{P}^i[\cdot]$ the probabilities in each possibility $i \in [K]$; likewise for the expectations $\mathbb{E}^*[\cdot]$ or $\mathbb{E}^i[\cdot]$.

Lemma 11 lower-bounds the query complexity of an identification algorithm $\mathcal{B}$.

**Lemma 11** (Identification Lower Bounds)**.** *To identify the hard instances $\{\boldsymbol{F}^i\}_{i\in[K]}$ each with probability $\geq \frac{2}{3}$, an identification algorithm $\mathcal{B}$ makes at least $\mathbb{E}^*[T] = \Omega(\varepsilon^{-3})$ pricing queries on the base instance $\boldsymbol{F}^*$ (in expectation over the randomness of $\mathcal{B}$ itself and $\boldsymbol{F}^*$).*

Proof. Consider the base instance $\boldsymbol{F}^*$ and a specific hard instance $\boldsymbol{F}^i$. As mentioned, only pricing queries within the corresponding modification interval $(s^i, t^i]$ can help identify $\boldsymbol{F}^i$.

Recall that $\mathsf{KL}(p, q) = p\ln\left(\frac{p}{q}\right) + (1-p)\ln\left(\frac{1-p}{1-q}\right)$ denotes the KL divergence between two Bernoulli distributions with parameters $p, q \in [0, 1]$. For $x \in (s^i, t^i] \subseteq [\frac{1}{2}, 1]$, we have $0 \leq F^*(x) - F^i(x) \leq \frac{4}{3}\varepsilon \leq \frac{1}{12}$ (Lemma 9) and $F^*(x) = \frac{3x-1}{3x} \in [\frac{1}{3}, \frac{2}{3}]$, so Claim 5 is applicable and gives

$$\mathsf{KL}\big(F^*(x),\ F^i(x)\big) \;\leq\; 3 \cdot (\tfrac{4}{3}\varepsilon)^2 \;=\; \tfrac{16}{3}\varepsilon^2.$$

Then, regarding the event $\mathscr{E}^i := \{\mathcal{B} \text{ outputs } \boldsymbol{F}^i\}$, we know from Claim 6 that

$$\mathsf{KL}\big(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i]\big) \leq \mathsf{KL}\big(F^*(x),\ F^i(x)\big) \cdot \mathbb{E}^*[T^i] \leq \tfrac{16}{3}\varepsilon^2 \cdot \mathbb{E}^*[T^i].$$

By enumerating all $i \in [K]$, we can upper-bound the sum $\sum_{i\in[K]} \mathsf{KL}(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i])$ as follows:

$$\textstyle\sum_{i\in[K]} \mathsf{KL}\big(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i]\big) \leq \sum_{i\in[K]} \frac{16}{3}\varepsilon^2 \cdot \mathbb{E}^*[T^i] \leq \frac{16}{3}\varepsilon^2 \cdot \mathbb{E}^*[T].$$

The last step uses the linearity of expectations and $\sum_{i\in[K]} T_i \leq T$ (in any possible outcome).

Moreover, since the KL divergence $\mathsf{KL}(p, q)$ is a convex function (Claim 4), using Jensen's inequality (Claim 3), we can lower-bound the sum $\sum_{i\in[K]} \mathsf{KL}(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i])$ as follows:

$$\begin{aligned}\textstyle\sum_{i\in[K]} \mathsf{KL}\big(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i]\big) &\geq K \cdot \mathsf{KL}\Big(\tfrac{\sum_{i\in[K]} \mathbb{P}^*[\mathscr{E}^i]}{K},\ \tfrac{\sum_{i\in[K]} \mathbb{P}^i[\mathscr{E}^i]}{K}\Big) \\ &\geq K \cdot \mathsf{KL}(\tfrac{1}{8}, \tfrac{2}{3}) \geq \tfrac{1}{2}K.\end{aligned}$$

The second step uses "$\{\mathscr{E}^i\}_{i\in[K]}$ are disjoint" $\Rightarrow \frac{\sum_{i\in[K]} \mathbb{P}^*[\mathscr{E}^i]}{K} \leq \frac{1}{K} \leq \frac{1}{8}$ and the premise of the lemma "$\mathbb{P}^i[\mathscr{E}^i] \geq \frac{2}{3}$ for $i \in [K]$". And the last step uses $\mathsf{KL}(\frac{1}{8}, \frac{2}{3}) \approx 0.6352$.

Combining the above two equations directly gives $\mathbb{E}^*[T] \geq \frac{3}{32}\varepsilon^{-2} \cdot K \geq \frac{21}{512}\varepsilon^{-3}$, where the last step uses $\varepsilon \in (0, \frac{1}{16}) \Rightarrow K = \lfloor \frac{1}{2}\varepsilon^{-1} \rfloor \geq \frac{7}{16}\varepsilon^{-1}$. This finishes the proof of Lemma 11. □

Finally, we can translate the query complexity lower bound of an identification algorithm (Lemma 11) into that of a pricing algorithm (Theorem 7).

Proof of Theorem 7. Combine the "pricing-to-identification" reduction and Lemma 11: if a pricing algorithm $\mathcal{A}$ always outputs a $\frac{1}{20}\varepsilon$-approximately optimal price $p^{\mathcal{A}}$ with probability $\geq \frac{2}{3}$, then it makes at least $\Omega(\varepsilon^{-3})$ pricing queries on the base instance $\boldsymbol{F}^*$.

Scaling the parameter $\varepsilon \in (0, \frac{1}{16})$ by a factor of $\frac{1}{20}$ finishes the proof of Theorem 7. □

## 4 $\Omega(\varepsilon^{-3})$ Lower Bound for Three MHR Distributions

In this section, we investigate the query complexity of Uniform Pricing in the setting with *MHR* distributions. Specifically, we will establish Theorem 12.

**Theorem 12.** *For three (or more) MHR distributions, the query complexity of Uniform Pricing is $\Omega(\varepsilon^{-3})$.*

*Remark.* This result is most relevant to the work [11] (as well as Theorem 7), which shows that:
(i) For *any number of general distributions*, the query complexity of Uniform Pricing is $\widetilde{\Theta}(\varepsilon^{-3})$.
(ii) For *a single MHR distribution*, the query complexity of Uniform Pricing is $\widetilde{\Theta}(\varepsilon^{-2})$.
Therefore, Theorem 12 complements [11] (as well as Theorem 7) by showing a more thorough picture: "a single MHR distribution" is a quite singular case — even a *minor generalization* to "three MHR distributions" will increase the query complexity from $\widetilde{\Theta}(\varepsilon^{-2})$ to the general-case bound $\widetilde{\Theta}(\varepsilon^{-3})$.

We are left with the intermediate case "two MHR distributions". Unfortunately, for this case we can only establish a query complexity lower bound of $\Omega(\varepsilon^{-5/2})$, which is deferred to the full version. It is interesting to close the gap between the best known bounds $\Omega(\varepsilon^{-5/2})$ and $O(\varepsilon^{-3})$ for this case.

In the remainder of this section, we will establish Theorem 12. Again, we consider a sufficiently small $\varepsilon \in (0, \frac{1}{48})$ and a sufficiently large $K := \lfloor \frac{1}{8}\varepsilon^{-1} \rfloor \geq 6$ throughout. The proof naturally adapts the techniques in Section 3 (from the "two regular distributions" case) to the "three MHR distributions" case. Firstly, we present in Section 4.1 the counterpart lower-bound construction, including one base instance $\boldsymbol{F}^*$ and $K$ hard instances $\{\boldsymbol{F}^i\}_{i\in[K]}$. Afterward, we present in Section 4.2 the counterpart lower-bound analysis.

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[9] I.e., this directly follows from a combination of Lemma 10, that the modification intervals $(s^i, t^i]$ are disjoint, and $\mathcal{A}$'s performance guarantees.

Without ambiguity, we often reload the notations introduced in Section 3.

## 4.1 Lower Bound Construction

Consider two parameters $t \in [\frac{7}{8} + \varepsilon, 1]$ and $s = s(t) := t - \varepsilon$. We define three parametric CDF's $F_1^{*t}$, $F_2^{*t}$, and $F_3^{*t}$:

$$F_1^{*t}(x) \;:=\; 1 - (\tfrac{3}{4})^x, \qquad\qquad x \in [0, 1],$$

$$F_2^{*t}(x) \;:=\; \begin{cases} \frac{1-\frac{3}{4t}}{1-(\frac{3}{4})^t} \cdot \frac{x}{t}, & x \in [0, t] \\ \frac{1-\frac{3}{4x}}{1-(\frac{3}{4})^x}, & x \in (t, 1] \end{cases},$$

$$F_3^{*t}(x) \;:=\; \begin{cases} 0 & x \in [0, \frac{3}{4}] \\ \frac{1-\frac{3}{4x}}{1-(\frac{3}{4})^x} \cdot \frac{1-(\frac{3}{4})^t}{1-\frac{3}{4t}} \cdot \frac{t}{x}, & x \in (\frac{3}{4}, t] \\ 1, & x \in (t, 1] \end{cases}.$$

(We will verify the MHR condition for these distributions later in Lemma 13.) In regard to Uniform Pricing, the first-order CDF $F^{*t}(x) := F_1^{*t}(x) \cdot F_2^{*t}(x) \cdot F_3^{*t}(x)$ and the revenue function $R^{*t}(x) := x \cdot (1 - F^{*t}(x))$ are given as follows:

$$F^{*t}(x) \;=\; \begin{cases} 0, & x \in [0, \frac{3}{4}] \\ \frac{x - \frac{3}{4}}{x}, & x \in (\frac{3}{4}, 1] \end{cases}, \qquad R^{*t}(x) \;=\; \begin{cases} x, & x \in [0, \frac{3}{4}] \\ \frac{3}{4}, & x \in (\frac{3}{4}, 1] \end{cases}.$$

These formulae $F^{*t}$ and $R^{*t}$ are independent of $t \in [\frac{7}{8} + \varepsilon, 1]$; accordingly, we can define our base instance $\boldsymbol{F}^* := F_1^{*t} \otimes F_2^{*t} \otimes F_3^{*t}$ using any specific $t \in [\frac{7}{8} + \varepsilon, 1]$.

To establish the desired lower bound $\Omega(\varepsilon^{-3})$, we modify the third parametric CDF $F_3^{*t}$ into another parametric CDF $F_3^t$ on the interval $(s, t] = (t - \varepsilon, t]$, as follows. (In contrast, the first and the second parametric CDF's $F_1^{*t}$ and $F_2^{*t}$ keep the same.)

$$F_3^t(x) \;:=\; \begin{cases} 1 - (1 - F_3^{*t}(s)) \cdot e^{-\frac{f_3^{*t}(s)}{1-F_3^{*t}(s)} \cdot (x-s)} & x \in (s, t] \\ \qquad = 1 - (1 - F_3^{*t}(s)) \cdot e^{-h_3^{*t}(s) \cdot (x-s)}, & \\ F_3^{*t}(x), & x \notin (s, t] \end{cases}.$$

(I.e., only pricing queries within the modification interval $(s, t]$ can help identify $F_3^t$.) Then, the modified first-order CDF $F^t(x) := F_1^{*t}(x) \cdot F_2^{*t}(x) \cdot F_3^t(x)$ and the modified revenue function $R^t(x) := x \cdot (1 - F^t(x))$ follow accordingly.

*Remark.* The modified parametric CDF $F_3^t$ is defined such that *"over the interval $x \in (s, t]$, the corresponding hazard rate function $h_3^t(x)$ is constant $= h_3^{*t}(s)$"*; below, the proof of Lemma 13 will show this explicitly. This induces an ODE, and solving it under the boundary condition $\lim_{x \to s^+} F_3^t(x) = F_3^{*t}(s)$ (so as to preserve the continuity at $x = s$) gives the above defining formula.

Lemma 13 verifies the MHR condition for all considered CDF's; we defer its proof to Section A.

**Lemma 13** (MHR). *Given any $t \in [\frac{7}{8} + \varepsilon, 1]$, all of $F_1^{*t}$, $F_2^{*t}$, $F_3^{*t}$, and $F_3^t$ are well-defined MHR CDF's.*

Lemmas 14 and 15 will be useful for our lower-bound analysis in Section 4.2; we defer their proofs to Section A.

**Lemma 14** (First-Order CDF's). *$0 \le F^{*t}(x) - F^t(x) \le \varepsilon$ for $x \in (s, t]$, while $F^{*t}(x) = F^t(x)$ for $x \notin (s, t]$.*

**Lemma 15** (Revenue Functions). *$R^t(t) \ge \frac{3}{4} + \frac{1}{60}\varepsilon$, while $R^t(x) \le \frac{3}{4}$ for $x \notin (s, t]$.*

To get the desired lower bound $\Omega(\varepsilon^{-3})$, we choose a sequence of *disjoint* modification intervals[10] $(s^i, t^i] := (\frac{7}{8} + i\varepsilon - \varepsilon, \frac{7}{8} + i\varepsilon]$ and, thus, obtain our hard instances $\boldsymbol{F}^i := F_1^{*t^i} \otimes F_2^{*t^i} \otimes F_3^{t^i}$ for $i \in [K]$.

## 4.2 Lower Bound Analysis

Consider a specific pricing algorithm $\mathcal{A}$ that performs well on all hard instances $\{\boldsymbol{F}^i\}_{i \in [K]}$: in any possibility $i \in [K]$, it always outputs a $\frac{1}{70}\varepsilon$-approximately optimal price $p^{\mathcal{A}}$ with probability $\ge \frac{2}{3}$:

$$R^i(p^{\mathcal{A}}) \;\ge\; \max_{p \in [0,1]} R^i(p) - \tfrac{1}{70}\varepsilon.$$

(In contrast, no performance guarantee for the base instance $\boldsymbol{F}^*$ is needed.) Based on this, we can develop a "pricing-to-identification" reduction for all base/hard instances $\{\boldsymbol{F}^*\} \cup \{\boldsymbol{F}^i\}_{i \in [K]}$, i.e., another identification algorithm $\mathcal{B}^{\mathcal{A}}$ with *exactly the same number of pricing queries*:

- Run $\mathcal{A}$ on an unknown instance $\boldsymbol{F}$ (promised to be one of $\{\boldsymbol{F}^*\} \cup \{\boldsymbol{F}^i\}_{i \in [K]}$), getting a price $p^{\mathcal{A}}$.
- If $p^{\mathcal{A}} \in (s^i, t^i]$ for some $i \in [K]$, output $\boldsymbol{F}^i$;
- Otherwise, output $\boldsymbol{F}^*$.

Clearly, $\mathcal{B}^{\mathcal{A}}$ can identify all hard instances $\{\boldsymbol{F}^i\}_{i \in [K]}$:[11] in any possibility $i \in [K]$, it always successfully outputs $\boldsymbol{F}^i$ with probability $\ge \frac{2}{3}$. (Again, no performance guarantee for the base instance $\boldsymbol{F}^*$ is needed.)

In the rest of this section, we consider a specific identification algorithm $\mathcal{B}$; denote by $T$ the number of pricing queries it makes and, in particular, $T^i$ for $i \in [K]$ the number of pricing queries it makes within the index-$i$ modification interval $(s^i, t^i]$. Also, denote by $\mathbb{P}^*[\cdot]$ or $\mathbb{P}^i[\cdot]$ the probabilities in each possibility $i \in [K]$; likewise for the expectations $\mathbb{E}^*[\cdot]$ or $\mathbb{E}^i[\cdot]$.

Lemma 16 lower-bounds the query complexity of an identification algorithm $\mathcal{B}$.

**Lemma 16** (Identification Lower Bounds). *To identify the hard instances $\{\boldsymbol{F}^i\}_{i \in [K]}$ each with probability $\ge \frac{2}{3}$, an identification algorithm $\mathcal{B}$ makes at least $\mathbb{E}^*[T] = \Omega(\varepsilon^{-3})$ pricing queries on the base instance $\boldsymbol{F}^*$ (in expectation over the randomness of $\mathcal{B}$ itself and $\boldsymbol{F}^*$).*

Finally, we can translate the query complexity lower bound of an identification algorithm (Lemma 16) into that of a pricing algorithm (Theorem 12).

Proof of Theorem 12. Combine the "pricing-to-identification" reduction and Lemma 16: if a pricing algorithm $\mathcal{A}$ always outputs a $\frac{1}{70}\varepsilon$-approximately optimal price $p^{\mathcal{A}}$ with probability $\ge \frac{2}{3}$, then it makes at least $\Omega(\varepsilon^{-3})$ pricing queries on the base instance $\boldsymbol{F}^*$.

Scaling the parameter $\varepsilon \in (0, \frac{1}{48})$ by a factor of $\frac{1}{70}$ finishes the proof of Theorem 12. □

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[10] Since $K = \lfloor \frac{1}{8}\varepsilon^{-1} \rfloor$, all parameters $\{t^i\}_{i \in [K]}$ belong to the interval $[\frac{7}{8} + \varepsilon, 1]$, satisfying the premises of Lemmas 13 and 15.

[11] I.e., this directly follows from a combination of Lemma 15, that the modification intervals $(s^i, t^i]$ are disjoint, and $\mathcal{A}$'s performance guarantees.

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## A Missing Proofs in Section 4

**Lemma 13** (MHR). *Given any $t \in [\frac{7}{8} + \varepsilon, 1]$, all of $F_1^{*t}$, $F_2^{*t}$, $F_3^{*t}$, and $F_3^t$ are well-defined MHR CDF's.*

Proof. Let $a := \ln(\frac{4}{3}) \approx 0.2877$ and $b = b(t) := \frac{1-(3/4)^t}{t-3/4} \cdot t^2$ for notational brevity.

The first function $F_1^{*t}(x) = 1 - (\frac{3}{4})^x$ for $x \in [0, 1]$ is clearly a well-defined MHR CDF.

The second function $F_2^{*t}$ is *continuous* at $x = t$ (since $\lim_{x \to t^+} F_2^{*t}(x) = \frac{t}{b} = F_2^{*t}(t)$), has a *positive* derivative function $f_2^{*t}$ (Claim 17), takes values between $0 = F_2^{*t}(0) \le F_2^{*t}(1) = 1$, and has a *increasing* hazard rate function $h_2^{*t}$ (Claim 18). Given these, $F_2^{*t}$ is a well-defined MHR CDF.

$$f_2^{*t}(x) = F_2^{*t\prime}(x) = \begin{cases} \frac{1}{b}, & x \in [0, t] \\ \frac{(\frac{4}{3})^x - 1 - ax \cdot (\frac{4}{3}x - 1)}{(1-(\frac{3}{4})^x)^2} \cdot \frac{3}{4x^2} \cdot (\frac{3}{4})^x, & x \in (t, 1] \end{cases},$$

$$h_2^{*t}(x) = \frac{f_2^{*t}(x)}{1-F_2^{*t}(x)} = \begin{cases} \frac{1}{b-x}, & x \in [0, t] \\ \frac{(\frac{4}{3})^x - 1 - ax \cdot (\frac{4}{3}x - 1)}{(1-(\frac{3}{4})^x) \cdot (\frac{3}{4x} - (\frac{3}{4})^x)} \cdot \frac{3}{4x^2} \cdot (\frac{3}{4})^x, & x \in (t, 1] \end{cases}.$$

**Claim 17.** $f_2^{*t}(x) \ge 0$ *for* $x \in [0, 1]$.

Proof. For $x \in (t, 1]$, note that $f_2^{*t}(x) \ge 0 \Leftrightarrow y(x) := (\frac{4}{3})^x - 1 - ax \cdot (\frac{4}{3}x - 1) \ge 0$. Even on the wider interval $[\frac{3}{4}, 1] \supseteq (t, 1]$, this function is *concave* $y''(x) = (\frac{4}{3})^x \cdot a^2 - \frac{8}{3}a \le \frac{4}{3}a^2 - \frac{8}{3}a \approx -0.6568 < 0$ and have *positive* endpoints $y(\frac{3}{4}) = (\frac{4}{3})^{3/4} - 1 \approx 0.2408$ and $y(1) = \frac{1-a}{3} \approx 0.2374$, which gives $y(x) \ge 0$ for $x \in [\frac{3}{4}, 1]$.

For $x \in [0, t]$, we trivially have $f_2^{*t}(x) = \frac{1}{b} \ge 0$. □

**Claim 18.** $h_2^{*t\prime}(x) \ge 0$ *for* $x \in [0, 1]$.

Proof. For $x \in (t, 1]$, we have $h_2^{*t}(x) = \frac{N(x)}{D(x)}$, where

$$\begin{aligned} N(x) &:= \frac{3}{4x^2} - (a - \frac{3a}{4x} + \frac{3}{4x^2}) \cdot (\frac{3}{4})^x, \\ D(x) &:= \frac{3}{4x} - (1 + \frac{3}{4x}) \cdot (\frac{3}{4})^x + (\frac{3}{4})^{2x}. \end{aligned}$$

Note that $h_2^{*t\prime}(x) \ge 0 \Leftrightarrow y(x) := N'(x) \cdot D(x) - N(x) \cdot D'(x) \ge 0$; we would even prove this on the wider interval $x \in [\frac{3}{4}, 1]$. By elementary algebra, we deduce that

$$\begin{aligned} N'(x) &= -\frac{3}{2x^3} + (a^2 - \frac{3a^2}{4x} + \frac{3}{2x^3}) \cdot (\frac{3}{4})^x, \\ D'(x) &= -\frac{3}{4x^2} + (a + \frac{3a}{4x} + \frac{3}{4x^2}) \cdot (\frac{3}{4})^x - 2a \cdot (\frac{3}{4})^{2x}, \\ y(x) &= (\frac{3}{4})^x \cdot \Big(a^2 \cdot (1 - \frac{3}{4x}) \cdot (\frac{3}{4x} - (\frac{3}{4})^{2x}) \\ &\qquad + \frac{9}{16x^4} \cdot (1 - (\frac{3}{4})^x)^2 \cdot (\frac{8x \cdot (1-ax)}{3} - (\frac{4}{3})^x)\Big). \end{aligned}$$

For $x \in [\frac{3}{4}, 1]$, it is easy to check $\big(1 \ge \frac{3}{4x}\big) \wedge \big(\frac{3}{4x} \ge (\frac{3}{4})^{2x}\big) \wedge \big(\frac{8x \cdot (1-ax)}{3} \ge \frac{5x}{3} \ge (\frac{4}{3})^x\big)$, which gives $y(x) \ge 0$.

For $x \in [0, t]$, we trivially have $h_2^{*t\prime}(x) = \frac{1}{(b-x)^2} \ge 0$.

At the division point $x = t \in [\frac{7}{8} + \varepsilon, 1]$, $\lim_{x \to t^+} h_2^{*t}(x) \ge h_2^{*t}(t)$ can be deduced as follows:

$$\begin{aligned} & \lim_{x \to t^+} h_2^{*t}(x) \ \ge \ h_2^{*t}(t) \\ \Leftrightarrow \quad & \frac{\frac{3}{4t^2} \cdot (1-(\frac{3}{4})^t) - (1-\frac{3}{4t}) \cdot (\frac{3}{4})^t \cdot a}{(1-(\frac{3}{4})^t) \cdot (\frac{3}{4t} - (\frac{3}{4})^t)} \ \ge \ \frac{1}{b-t} \ \equiv \ \frac{1-\frac{3}{4t}}{(\frac{3}{4t} - (\frac{3}{4})^t) \cdot t} \\ \Leftrightarrow \quad & (\frac{2}{3} - at) \cdot (t - \frac{3}{4}) + \big((\frac{3}{2} - t) \cdot (\frac{4}{3})^t - 1 + \frac{t}{3}\big) \ \ge \ 0. \end{aligned}$$

Even on the wider interval $t \in [\frac{7}{8}, 1]$, it is easy to check $\big(\frac{2}{3} \ge at\big) \wedge \big(t \ge \frac{3}{4}\big) \wedge \big((\frac{3}{2} - t) \cdot (\frac{4}{3})^t - 1 + \frac{t}{3} \ge 0\big)$, which gives $\lim_{x \to t^+} h_2^{*t}(x) \ge h_2^{*t}(t)$. This finishes the proof of Claim 18. □

The third function $F_3^{*t}$ is $(\frac{3}{4}, t]$-supported, is *continuous* at both endpoints $\lim_{x \to (\frac{3}{4})^+} F_3^{*t}(x) = 0 = F_3^{*t}(\frac{3}{4})$ and $\lim_{x \to t^+} F_3^{*t}(x) = 1 = F_3^{*t}(t)$, has a *positive* derivative function $f_3^{*t} = F_3^{*t\prime}(x)$ (Claim 19), takes values between $0 = F_3^{*t}(\frac{3}{4}) \le F_3^{*t}(t) = 1$, and has a *increasing* hazard rate function $h_3^{*t} = \frac{f_3^{*t}(x)}{1-F_3^{*t}(x)}$ (Claim 20). Given these, $F_3^{*t}$ is a well-defined MHR CDF.

$$\begin{aligned} f_3^{*t}(x) &= \frac{(6-4x) - (\frac{3}{4})^x \cdot (4a \cdot x^2 - (3a+4) \cdot x + 6)}{(1-(\frac{3}{4})^x)^2} \cdot \frac{b}{4x^3}, & x \in (\tfrac{3}{4}, t], \\ h_3^{*t}(x) &= \frac{(6-4x) - (\frac{3}{4})^x \cdot (4a \cdot x^2 - (3a+4) \cdot x + 6)}{(\frac{1}{b} \cdot (1-(\frac{3}{4})^x) - \frac{4x-3}{4x^2}) \cdot (1-(\frac{3}{4})^x)} \cdot \frac{1}{4x^3}, & x \in (\tfrac{3}{4}, t]. \end{aligned}$$

**Claim 19.** $f_3^{*t}(x) \ge 0$ *for* $x \in (\frac{3}{4}, t]$.

Proof. By elementary algebra, $f_3^{*t}(x) \ge 0 \Leftrightarrow y(x) \ge 0$.

$$y(x) := (6 - 4x) - (\tfrac{3}{4})^x \cdot \big(4a \cdot x^2 - (3a + 4) \cdot x + 6\big).$$

By elementary algebra, we have

$$y'(x) = -4 + (\tfrac{3}{4})^x \cdot \big(4a^2 \cdot x^2 - (3a^2 + 12a) \cdot x + (9a + 4)\big).$$

Consider the parabola $z(x) := 4a^2 \cdot x^2 - (3a^2 + 12a) \cdot x + (9a + 4)$; it opens upward (since $4a^2 > 0$) and has an axis of symmetry $x = \frac{12+3a}{8a} \approx 5.5888 > 1$. Thus, even on the wider interval $x \in [\frac{3}{4}, 1]$, it is *decreasing* and *positive* $z(x) \geq z(1) = a^2 - 3a + 4 \approx 3.2197 > 0$. Even on the wider interval $x \in [\frac{3}{4}, 1]$, it is easy to see that $4a^2 \cdot x^2 - (3a^2 + 12a) \cdot x + (9a+4) \leq 4a^2 \cdot 1^2 - (3a^2 + 12a) \cdot \frac{3}{4} + (9a+4) \approx 4.1448 < \frac{9}{2} \Rightarrow y'(x) \leq -4 + (\frac{3}{4})^{3/4} \cdot \frac{9}{2} \approx -0.3733$ and, consequently, that $y(x)$ is *decreasing* and *positive* $y(x) \geq y(1) = \frac{1}{2} - \frac{3}{4}a \approx 0.2842$.

This finishes the proof of Claim 19. □

**Claim 20.** $h_3^{*t\prime}(x) \geq 0$ *for* $x \in (\frac{3}{4}, t]$.

Proof. For $x \in (\frac{3}{4}, t]$, we have $h_3^{*t}(x) = \frac{N(x)}{D(x)}$, where

$$\begin{aligned} N(x) &:= \Big(\tfrac{3-2x}{2x^3} - \tfrac{6-(4+3a)\cdot x+4ax^2}{4x^3} \cdot (\tfrac{3}{4})^x\Big) \cdot \tfrac{x}{t-x}, \\ D(x) &:= \Big(\tfrac{1}{b} \cdot (1 - (\tfrac{3}{4})^x) - \tfrac{4x-3}{4x^2}\Big) \cdot (1 - (\tfrac{3}{4})^x) \cdot \tfrac{x}{t-x}. \end{aligned}$$

Firstly, we assert that $N(x)$ is an *increasing* and *positive* function on $x \in (\frac{3}{4}, t]$. By elementary algebra,

$$\begin{aligned} N'(x) &= (4x-3) \cdot \tfrac{(4-2x)\cdot(1-(\frac{3}{4})^x)-ax\cdot(\frac{3}{4})^x}{4x^3\cdot(t-x)^2} \\ &\quad + (4x-3) \cdot \tfrac{a^2x^2\cdot(1-x)\cdot(\frac{3}{4})^x}{4x^3\cdot(t-x)^2} \\ &\quad + (1-t) \cdot \tfrac{(12-4x)\cdot(1-(\frac{3}{4})^x)+ax\cdot(4x-3)\cdot(1-ax)\cdot(\frac{3}{4})^x}{4x^3\cdot(t-x)^2}. \end{aligned}$$

Here each of these three summands is *positive*, given that $t \in [\frac{7}{8} + \varepsilon, 1]$, $x \in (\frac{3}{4}, t]$, and $a = \ln(\frac{4}{3}) \approx 0.2877$; for the first summand specifically, even on the wider interval $x \in [0, 1]$, we have $(\frac{3}{4})^x \leq 1 - \frac{x}{4} \leq 1 \Rightarrow (4 - 2x) \cdot (1 - (\frac{3}{4})^x) - ax \cdot (\frac{3}{4})^x \geq (4 - 2x) \cdot \frac{x}{4} - ax \geq (4 - 2x) \cdot \frac{x}{4} - \frac{x}{2} = \frac{x\cdot(1-x)}{2} \geq 0$.

The monotonicity of $N(x)$ together with $N(\frac{3}{4}) = \frac{4/3-(4/3)^{1/4}}{t-3/4} \geq 0$ gives our assertion.

Secondly, we assert that $D(x)$ is a *decreasing* and *positive* function on $x \in (\frac{3}{4}, t]$. By elementary algebra,

$$\begin{aligned} D'(x) &= \tfrac{y(x)}{(t-x)^2}, \\ y(x) &:= (\tfrac{t}{b} + \tfrac{6x-3t}{4x^2} - 1) \\ &\quad - \Big(a \cdot (x - \tfrac{3}{4}) \cdot \tfrac{t-x}{x} + \tfrac{6x-3t}{4x^2} - 1\Big) \cdot (\tfrac{3}{4})^x \\ &\quad - \Big((t - 2ax \cdot (t-x)) \cdot (1 - (\tfrac{3}{4})^x) + t\Big) \cdot \tfrac{1}{b} \cdot (\tfrac{3}{4})^x. \end{aligned}$$

The function $y(x)$ is *increasing* on the interval $x \in (\frac{3}{4}, t]$; by elementary algebra,

$$\begin{aligned} y'(x) &= \Big(\tfrac{3}{2x^2} \cdot \big((\tfrac{4}{3})^x - (1 + ax)\big) + a^2 \cdot (x - \tfrac{3}{4})\Big) \cdot \tfrac{t-x}{x} \cdot (\tfrac{3}{4})^x \\ &\quad + \Big(2 \cdot (1 - (\tfrac{3}{4})^x) \cdot (1 - ax) + ax\Big) \cdot \tfrac{2a}{b} \cdot (t-x) \cdot (\tfrac{3}{4})^x \geq 0. \end{aligned}$$

The last step uses $t \in [\frac{7}{8} + \varepsilon, 1]$, $a = \ln(\frac{4}{3}) \approx 0.2877$, and $(\frac{4}{3})^x \geq 1 + ax$.

The monotonicity of $y(x)$ together with $y(t) = 0$ (elementary algebra) implies $D'(x) \leq 0$, for $x \in (\frac{3}{4}, t]$. This in combination with $D(t) = 0$ (elementary algebra) gives our assertion.

Clearly, combining both assertions finishes the proof of Claim 20. □

The modified function $F_3^t$ is $(\frac{3}{4}, t]$-supported — with different defining formulae on the sub-intervals $(\frac{3}{4}, s]$ and $(s, t]$ — is *continuous* at the division point $\lim_{x \to s^+} F_3^t(x) = F_3^{*t}(s) = F_3^t(s)$, has a *positive* derivative function $f_3^t = F_3^{t\prime}(x)$ (as follows), takes values between $0 = F_3^t(\frac{3}{4}) \leq F_3^t(t) \leq 1$, and has a *increasing* hazard rate function $h_3^t = \frac{f_3^t(x)}{1-F_3^t(x)}$ (as follows). Given these, $F_3^t$ is a well-defined MHR CDF.

$$f_3^t(x) = \begin{cases} f_3^{*t}(s) \cdot e^{-\frac{f_3^{*t}(s)}{1-F_3^{*t}(s)} \cdot (x-s)}, & x \in (s, t] \\ \quad = f_3^{*t}(s) \cdot e^{-h_3^{*t}(s) \cdot (x-s)} & \\ f_3^{*t}(x), & x \notin (s, t] \end{cases},$$

$$h_3^t(x) = \begin{cases} \frac{f_3^{*t}(s) \cdot e^{-h_3^{*t}(s)\cdot(x-s)}}{(1-F_3^{*t}(s)) \cdot e^{-h_3^{*t}(s)\cdot(x-s)}} = h_3^{*t}(s) & x \in (s, t] \\ h_3^{*t}(x), & x \notin (s, t] \end{cases}.$$

This finishes the proof of Lemma 13. □

**Lemma 14** (First-Order CDF's). $0 \leq F^{*t}(x) - F^t(x) \leq \varepsilon$ *for* $x \in (s, t]$, *while* $F^{*t}(x) = F^t(x)$ *for* $x \notin (s, t]$.

Proof. Following the proof of Lemma 13, we have $h_3^{*t}(x) \geq h_3^{*t}(s) = h_3^t(x) \Leftrightarrow -\frac{f_3^{*t}(x)}{1-F_3^{*t}(x)} \leq -\frac{f_3^t(x)}{1-F_3^t(x)} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln\big(1 - F_3^{*t}(x)\big) \leq \frac{\mathrm{d}}{\mathrm{d}x}\ln\big(1 - F_3^t(x)\big)$, for $x \in (s, t]$. Together with the boundary condition $F_3^{*t}(s) = F_3^t(s)$, we deduce that $F_3^{*t}(x) - F_3^t(x) \geq 0 \Rightarrow F^{*t}(x) - F^t(x) = F_1^{*t}(x) \cdot F_2^{*t}(x) \cdot \big(F_3^{*t}(x) - F_3^t(x)\big) \geq 0$, for $x \in (s, t]$. In addition, we have $f^{*t}(x) - f^t(x) \leq f^{*t}(x) = \frac{3}{4x^2} \leq 1$, for $x \in (s, t] \subseteq [\frac{7}{8}, 1]$, where the first step uses the modified first-order PDF $f^t$'s positivity. Together with the boundary condition $F^{*t}(s) = F^t(s)$, we deduce that $F^{*t}(x) - F^t(x) \leq 1 \cdot (x - s) \leq t - s = \varepsilon$, for $x \in (s, t]$.

The second part "$F^{*t}(x) = F^t(x)$ for $x \notin (s, t]$" is obvious. □

**Lemma 15** (Revenue Functions). $R^t(t) \geq \frac{3}{4} + \frac{1}{60}\varepsilon$, *while* $R^t(x) \leq \frac{3}{4}$ *for* $x \notin (s, t]$.

Proof. Note that $t \in [\frac{7}{8} + \varepsilon, 1]$, $s = t - \varepsilon$, $F_1^{*t}(t) \cdot F_2^{*t}(t) = 1 - \frac{3}{4t}$, $F_3^{*t}(t) = 1$, and $R^{*t}(t) = \frac{3}{4}$. Using the defining formulae of $R^t(t)$, $R^{*t}(t)$, and $F_3^t(t)$, we can deduce "$R^t(t) \geq \frac{3}{4} + \frac{1}{60}\varepsilon$" as follows:

$$\begin{aligned} R^t(t) - R^{*t}(t) &= t \cdot F_1^{*t}(t) \cdot F_2^{*t}(t) \cdot (F_3^{*t}(t) - F_3^{*t}(s)) \cdot e^{-h_3^{*t}(s)\cdot(t-s)} \\ &= (t - \tfrac{3}{4}) \cdot \Big(\tfrac{F_3^{*t}(t)-F_3^{*t}(s)}{t-s} \cdot \varepsilon\Big) \cdot e^{-h_3^{*t}(t-\varepsilon)\cdot\varepsilon} \\ &\geq (t - \tfrac{3}{4}) \cdot \big(f_3^{*t}(1) \cdot \varepsilon\big) \cdot e^{-h_3^{*t}(\frac{3}{4})\cdot(t-\frac{3}{4})} \\ &\geq (t - \tfrac{3}{4}) \cdot (b \cdot \varepsilon) \cdot e^{-9b\cdot(t-\frac{3}{4})} \geq \tfrac{1}{60}\varepsilon. \end{aligned}$$

Here the third step uses the mean value theorem and Claims 21 and 22. The fourth step uses $a = \ln(\frac{4}{3}) \approx 0.2877 \Rightarrow f_3^{*t}(1) = (2-3a)\cdot b \approx 1.1370\cdot b \geq b$ and $h_3^{*t}(\frac{3}{4}) = \frac{16/9}{1-(3/4)^{3/4}}\cdot b \approx 9.1604\cdot b \geq 9b$. And the last step uses $t \in [\frac{7}{8} + \varepsilon, 1] \Rightarrow b \cdot (t - \frac{3}{4}) = t^2 \cdot (1 - (\frac{3}{4})^t) \in [\frac{49\cdot(1-(3/4)^{7/8})}{64} \approx 0.1703, \frac{1}{4}] \subseteq [\frac{1}{6}, \frac{1}{4}]$, which implies that $b \cdot (t - \frac{3}{4}) \cdot e^{-9b\cdot(t-\frac{3}{4})} \geq \frac{1}{6}e^{-9/4} \approx 0.0176 \geq \frac{1}{60}$.

**Claim 21.** $f_3^{*t}(x)$ *is decreasing on* $x \in [\frac{3}{4}, 1]$.

Proof. Recall that $f_3^{*t}(x) = \frac{(6-4x)-(\frac{3}{4})^x\cdot(4a\cdot x^2-(3a+4)\cdot x+6)}{(1-(\frac{3}{4})^x)^2} \cdot \frac{b}{4x^3}$. By elementary algebra, $f_3^{*t\prime}(x) = -\frac{b}{4x^3\cdot(1-(\frac{3}{4})^x)^3} \cdot y(x)$, where $y(x) := 4a\cdot(\frac{3}{4})^x\cdot\underbrace{\big((3 - 2x) \cdot (1 - (\frac{3}{4})^x) - ax \cdot (x - \frac{3}{4}) \cdot (1 + (\frac{3}{4})^x)\big)}_{\heartsuit}$

$+\ \underbrace{(\tfrac{18}{x} - 8) \cdot (1 - (\tfrac{3}{4})^x)^2}_{\geq 0}$. Since $x \in [\tfrac{3}{4}, 1]$ and $a = \ln(\tfrac{4}{3}) \approx 0.2877$, we have $\heartsuit \geq (3-2) \cdot (1 - (\tfrac{3}{4})^{3/4}) - a \cdot \tfrac{1}{4} \cdot (1 + \tfrac{3}{4}) \approx 0.0682 > 0$, thus $y(x) \geq 0 \Rightarrow f_3^{*t\prime}(x) \leq 0$. This finishes the proof of Claim 21. $\square$

**Claim 22.** *$y(x) := x \cdot h_3^{*t}(t - x)$ is increasing on $x \in [0, t - \tfrac{3}{4}]$.*

$\triangleright$ *Note that $t \in [\tfrac{7}{8} + \varepsilon, 1] \Rightarrow \varepsilon \in [0, t - \tfrac{3}{4}]$.*

Proof. Recall that $h_3^{*t}(x) = \frac{f_3^{*t}(x)}{1 - F_3^{*t}(x)}$. By elementary algebra,

$$y'(x) \ = \ x \cdot h_3^{*t}(t - x) \cdot \tfrac{\mathrm{d}}{\mathrm{d}x}\big(\ln\big(x \cdot h_3^{*t}(t - x)\big)\big).$$

Let us substitute $z = (t - x) \in [\tfrac{3}{4}, t]$. Clearly, to show that $y(x)$ is *increasing* on $x \in [0, t - \tfrac{3}{4}]$, it suffices to show that, under the same $t \in [\tfrac{7}{8}, 1]$, the function $g(z, t)$ is *decreasing* on $z \in [\tfrac{3}{4}, t]$.

$$\begin{aligned} g(z,t) \ &:= \ (t - z) \cdot h_3^{*t}(z) \ = \ \tfrac{N(z,t)}{D(z,t)}, \\ N(z,t) \ &:= \ \tfrac{6 - 4z}{z \cdot (4 - 3z)} - a \cdot \tfrac{4z - 3}{((\frac{4}{3})^z - 1) \cdot (4 - 3z)}, \\ D(z,t) \ &:= \ \tfrac{(1 - (\frac{3}{4})^z) \cdot \frac{4z^2}{b} - (4z - 3)}{t - z} \cdot \tfrac{1}{4 - 3z}. \end{aligned}$$

Firstly, we assert that $N(z, t)$ is *decreasing* and *positive* $N(z, t) \geq N(1, t) = 2 - 3a \approx 1.1370$, even on the wider interval $z \in [\tfrac{3}{4}, 1]$. By rewriting $N(z, t) = N_1(z, t) - a \cdot \frac{N_2(z,t)}{N_3(z,t)}$, the monotonicity follows from a combination of three observations:

(i) The function $N_1(z, t) := \frac{6 - 4z}{z \cdot (4 - 3z)}$ is *decreasing* $\frac{\partial}{\partial z} N_1(z, t) = -12 \cdot \frac{(1 - z) \cdot (2 - z)}{z^2 \cdot (4 - 3z)^2} \leq 0$.

(ii) The function $N_2(z, t) := 4z - 3$ is *increasing* (obvious) and *positive* (obvious).

(iii) The function $N_3(z, t) := ((\tfrac{4}{3})^z - 1) \cdot (4 - 3z)$ is *decreasing* $\frac{\partial}{\partial z} N_3(z, t) = 3 \cdot (\tfrac{4}{3})^x \cdot ((\tfrac{3}{4})^x - ax - 1 + \tfrac{4}{3}a) \leq 0$, given that $(\tfrac{3}{4})^x - ax - 1 + \tfrac{4}{3}a \leq (\tfrac{3}{4})^{3/4} - \tfrac{3}{4}a - 1 + \tfrac{4}{3}a \approx -0.0263$, and *positive* (obvious).

Secondly, we assert that $D(z, t)$ is *increasing* and *positive* $D(z, t) \geq D(\tfrac{3}{4}, t) = \tfrac{9}{7} t^{-2} \cdot \frac{1 - (3/4)^{3/4}}{1 - (3/4)^t} \geq 0$, even on the wider interval $z \in [\tfrac{3}{4}, 1]$. To see the monotonicity, by elementary algebra,

$$\begin{aligned} \tfrac{\partial}{\partial z} D(z,t) \ &= \ \tfrac{1}{(t - z)^2 \cdot (4 - 3x)^2} \cdot \tfrac{4}{b} \cdot K^z(t), \\ K^z(t) \ &:= \ \big(z^2 \cdot (1 - (\tfrac{3}{4})^z) \cdot (4 - 3t) \\ &\qquad - t^2 \cdot (1 - (\frac{3}{4})^t) \cdot (4 - 3z) \cdot \frac{4z - 3}{4t - 3}\big) \\ &\quad + \ (t - z) \cdot z \cdot \big(az \cdot (4 - 3z) \cdot (\tfrac{3}{4})^z + 8 \cdot (1 - (\tfrac{3}{4})^z)\big) \\ &\quad - \ (t - z) \cdot (1 - (\tfrac{3}{4})^t) \cdot \tfrac{7t^2}{4t - 3}. \end{aligned}$$

Clearly, it suffices to show $K^z(t) \geq 0$ for $z, t \in [\tfrac{3}{4}, 1]$. Below, given $z \in [\tfrac{3}{4}, 1]$, let us analyze this function. By elementary algebra, its first-order derivative $K^{z\prime}(t)$ and third-order derivative $K^{z\prime\prime\prime}(t)$ are given by

$$\begin{aligned} K^{z\prime}(t) \ &= \ (8z - 3z^2) \cdot \big((\tfrac{3}{4})^t - (\tfrac{3}{4})^z\big) \\ &\quad + \ a \cdot \tfrac{4 - 3z}{4z - 3} \cdot \big(\tfrac{z^2}{4z - 3} \cdot (\tfrac{3}{4})^z - \tfrac{t^2}{4t - 3} \cdot (\tfrac{3}{4})^t\big) \\ &\quad + \ (t - z) \cdot \big(\tfrac{-56t^2 + 111t - 72 + 27z}{(4t - 3)^2} \cdot \big(1 - (\tfrac{3}{4})^t\big) - \tfrac{7at^2}{4t - 3} \cdot (\tfrac{3}{4})^t\big), \\ K^{z\prime\prime\prime}(t) \ &= \ -\tfrac{7a}{4} \cdot (\tfrac{3}{4})^t \cdot (a^2t^2 - 6at + 6) \\ &\quad - 162 \cdot \tfrac{(4u - 3)^2}{(4t - 3)^4} \cdot \big(1 - (\tfrac{3}{4})^t \cdot L(t)\big), \end{aligned}$$

$$\begin{aligned} L(t) \ &:= \ 1 + \tfrac{3}{4} \cdot (\tfrac{4}{3}t - 1) \cdot a + \tfrac{t \cdot (8t^2 - 14t + 9)}{12} \cdot (\tfrac{4}{3}t - 1)^2 \cdot a^2 \\ &\quad + \frac{t^2}{8} \cdot (\frac{4}{3}t - 1)^3 \cdot a^3. \end{aligned}$$

Also, we have $\big(a = \ln(\tfrac{4}{3}) \approx 0.2877\big) \wedge \big(t \in [\tfrac{3}{4}, 1]\big) \Rightarrow a^2t^2 - 6at + 6 \geq 0$ and "$L(t)$ is increasing on $t \in [\tfrac{3}{4}, 1]$" $\Rightarrow L(t) \leq L(1) = 1 + \frac{a}{4} + \frac{a^2}{36} + \frac{a^3}{216} \approx 1.0743 \leq (\tfrac{4}{3})^{3/4} \approx 1.2408 \leq (\tfrac{4}{3})^t$. Hence, on the interval $t \in [\tfrac{3}{4}, 1]$, the third-order derivative is *negative* $K^{z\prime\prime\prime}(t) \leq 0$ and the first-order derivative $K^{z\prime}(t)$ is *concave*.

At the point $t = z$, the second-order derivative $K^{z\prime\prime}(t)$ evaluates to $K^{z\prime\prime}(z) = M(z) \geq 0$, where

$$\begin{aligned} M(z) \ &:= \ \underbrace{\tfrac{24 - 14z}{4z - 3} \cdot (1 - (\tfrac{3}{4})^z)}_{\heartsuit} \ + \ \underbrace{\tfrac{a^2 \cdot (3/4)^z}{(4z - 3)^2} \cdot 9z^3 \cdot (\tfrac{4}{3} - z) \cdot (\tfrac{8}{3} - z)}_{\geq 0} \\ &\quad + \tfrac{a \cdot (3/4)^z}{(4z - 3)^3} \cdot \big((1 - z) \cdot \underbrace{(-156z^4 + 702z^3 - 598z^2 + 218z + 2)}_{\blacklozenge} - 2\big) \\ &\geq \ \tfrac{5}{2} + \tfrac{a \cdot (3/4)^z}{(4z - 3)^3} \cdot \big((1 - z) \cdot 168 \cdot (3z - 2) - 2\big) \\ &\geq \ \tfrac{5}{2} + \tfrac{1}{(4z - 3)^3} \cdot 36 \cdot (1 - z) \cdot (3z - 2) - \tfrac{1}{(4z - 3)^3} \cdot \tfrac{1}{2} \\ &= \ \tfrac{1}{(4z - 3)^3} \cdot \big(\tfrac{7}{4} + \underbrace{160 \cdot (z - \tfrac{3}{4}) \cdot (\tfrac{9}{8} - z) \cdot (\tfrac{21}{20} - z)}_{\geq 0}\big) \ \geq \ 0. \end{aligned}$$

Here the second step uses $\heartsuit \geq \frac{24 - 14z}{4z - 3} \cdot \frac{z}{4} \geq \frac{5}{2}$ and $\blacklozenge = 168 \cdot (3z - 2) + 26 \cdot (1 - z)^2 \cdot (-6z^2 + 15z + 13) \geq 168 \cdot (3z - 2)$ for $z \in [\tfrac{3}{4}, 1]$, both of which can be verified via elementary algebra. And the third step uses $a \cdot (3/4)^z \cdot 168 \geq 126a \approx 36.2479 \geq 36$ and $2a \cdot (3/4)^z \leq 2a \cdot (3/4)^{3/4} \approx 0.4637 \leq \frac{1}{2}$ for $z \in [\tfrac{3}{4}, 1]$.

At the point $t = z$, the first-order derivative $K^{z\prime}(t)$ equals zero $K^{z\prime}(z) = 0$ (obvious). Together with its *concavity* and that $K^{z\prime\prime}(z) = M(z) \geq 0$, this means that $K^z(t)$ is *decreasing* on the interval $t \in [\tfrac{3}{4}, z]$ and is *"either increasing or first-increasing-then-decreasing"* on the interval $t \in [z, 1]$; in any case, we always have

$$\min\nolimits_{t \in [\frac{3}{4}, 1]} K^z(t) \ = \ \min\big(K^z(z), \ K^z(1)\big).$$

The function $K^z(t)$ equals zero $K^z(z) = 0$ at the point $t = z$ (obvious). Accordingly, to show $K^z(t) \geq 0$ for $t \in [\tfrac{3}{4}, 1]$, it remains to show $K^z(1) \geq 0$, as follows.

$$\begin{aligned} K^z(1) \ &= \ (\tfrac{3}{4})^z \cdot z \cdot ((1 - z) \cdot az \cdot (4 - 3z) - (8 - 7z)) - \tfrac{(16z^2 - 14z - 5)}{4} \\ &= \ (\tfrac{3}{4})^z \cdot (1 - z) \cdot \big((a - \tfrac{1 - (3/4)^{1-z}}{1 - z}) + ((\tfrac{3}{4})^{1-z} - 1) \cdot \tfrac{16z + 2}{3} \\ &\qquad + (1 - z) \cdot (\frac{5}{3} + a \cdot (3z^2 - z - 1))\big) \\ &\geq \ (\tfrac{3}{4})^z \cdot (1 - z) \cdot \big(0 - a \cdot (1 - z) \cdot \tfrac{16z + 2}{3} \\ &\qquad + (1 - z) \cdot (\frac{5}{3} + a \cdot (3z^2 - z - 1))\big) \\ &= \ (\tfrac{3}{4})^z \cdot (1 - z)^2 \cdot ((\tfrac{5}{3} - 5a) + \tfrac{a}{3}(1 - z) + 3a(1 - z)^2) \ \geq \ 0. \end{aligned}$$

Here the second step rearranges the equation. The third step uses $\frac{1 - (3/4)^{1-z}}{1 - z} \leq a = \ln(\tfrac{4}{3}) \approx 0.2877$ for $z \in [\tfrac{3}{4}, 1]$. And the fourth step rearranges the equation. $\square$

The second part "$R^t(x) = R^{*t}(x) \leq \frac{3}{4}$ for $x \notin (s, t]$" is obvious. This finishes the proof of Lemma 15. $\square$

**Lemma 16** (Identification Lower Bounds)**.** *To identify the hard instances $\{\boldsymbol{F}^i\}_{i\in[K]}$ each with probability $\geq \frac{2}{3}$, an identification algorithm $\mathcal{B}$ makes at least $\mathbb{E}^*[T] = \Omega(\varepsilon^{-3})$ pricing queries on the base instance $\boldsymbol{F}^*$ (in expectation over the randomness of $\mathcal{B}$ itself and $\boldsymbol{F}^*$).*

Proof. Consider the base instance $\boldsymbol{F}^*$ and a specific hard instance $\boldsymbol{F}^i$. As mentioned, only pricing queries within the corresponding modification interval $(s^i, t^i]$ can help identify $\boldsymbol{F}^i$.

Recall that $\mathsf{KL}(p, q) = p\ln\left(\frac{p}{q}\right) + (1-p)\ln\left(\frac{1-p}{1-q}\right)$ denotes the KL divergence between two Bernoulli distributions with parameters $p, q \in [0, 1]$. For $x \in (s^i, t^i] \subseteq [\frac{7}{8}, 1]$, we have $0 \leq F^*(x) - F^i(x) \leq \varepsilon \leq \frac{1}{48}$ (Lemma 14) and $F^*(x) = 1 - \frac{3}{4x} \in [\frac{1}{7}, \frac{1}{4}]$, so Claim 5 is applicable and gives

$$\mathsf{KL}\big(F^*(x),\ F^i(x)\big) \ \leq\ 3\varepsilon^2.$$

Then, regarding the event $\mathscr{E}^i := \{\mathcal{B} \text{ outputs } \boldsymbol{F}^i\}$, we know from Claim 6 that

$$\mathsf{KL}\big(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i]\big) \leq \mathsf{KL}\big(F^*(x),\ F^i(x)\big) \cdot \mathbb{E}^*[T^i] \leq 3\varepsilon^2 \cdot \mathbb{E}^*[T^i].$$

By enumerating all $i \in [K]$, $\sum_{i\in[K]} \mathsf{KL}(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i])$ can be upper-bounded as follows:

$$\textstyle\sum_{i\in[K]} \mathsf{KL}\big(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i]\big) \leq \sum_{i\in[K]} 3\varepsilon^2 \cdot \mathbb{E}^*[T^i] \leq 3\varepsilon^2 \cdot \mathbb{E}^*[T].$$

The last step uses the linearity of expectations and $\sum_{i\in[K]} T_i \leq T$ (almost surely over all possible randomness).

Moreover, since the KL divergence $\mathsf{KL}(p, q)$ is a convex function (Claim 4), using Jensen's inequality (Claim 3), we can lower-bound the sum $\sum_{i\in[K]} \mathsf{KL}(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i])$ as follows:

$$\begin{aligned}\textstyle\sum_{i\in[K]} \mathsf{KL}\big(\mathbb{P}^*[\mathscr{E}^i],\ \mathbb{P}^i[\mathscr{E}^i]\big) &\geq K \cdot \mathsf{KL}\Big(\tfrac{\sum_{i\in[K]} \mathbb{P}^*[\mathscr{E}^i]}{K},\ \tfrac{\sum_{i\in[K]} \mathbb{P}^i[\mathscr{E}^i]}{K}\Big) \\ &\geq K \cdot \mathsf{KL}(\tfrac{1}{6}, \tfrac{2}{3}) \geq \tfrac{1}{2}K.\end{aligned}$$

The second step uses "$\{\mathscr{E}^i\}_{i\in[K]}$ are disjoint" $\Rightarrow$ $\frac{\sum_{i\in[K]} \mathbb{P}^*[\mathscr{E}^i]}{K} \leq \frac{1}{K} \leq \frac{1}{6}$ and the premise of the lemma "$\mathbb{P}^i[\mathscr{E}^i] \geq \frac{2}{3}$ for $i \in [K]$". And the last step uses $\mathsf{KL}(\frac{1}{6}, \frac{2}{3}) = \frac{5}{6}\ln 5 - \frac{7}{6}\ln 2 \approx 0.5325$.

Combining the above two equations gives $\mathbb{E}^*[T] \geq \frac{1}{6}\varepsilon^{-2} \cdot K \geq \frac{5}{288}\varepsilon^{-3}$, where the last step uses $\varepsilon \in (0, \frac{1}{48}) \Rightarrow K = \lfloor \frac{1}{8}\varepsilon^{-1} \rfloor \geq \frac{5}{48}\varepsilon^{-1}$. This finishes the proof of Lemma 16. □

## B Regret Lower Bounds

In this appendix, we depart from the pricing query complexity problem and study the *regret minimization* problem. Recall the *first-order value distribution* $F$ and the *revenue function* $R$; as before, we consider $[0, 1]$-supported value distributions $\boldsymbol{F}$, so the *optimal uniform price* $p^{\mathrm{opt}} = p^{\mathrm{opt}}(\boldsymbol{F}) = \mathrm{argmax}_{p\in[0,1]} R(p)$ is well-defined and lies in the support $[0, 1]$.

$$\begin{aligned} F(p) &= \mathbb{P}_{\boldsymbol{v}\sim\boldsymbol{F}}\big[(\max_{i\in[n]} v_i) < p\big] = \textstyle\prod_{i=1}^n F_i(p), && \forall p \geq 0, \\ R(p) &= p \cdot \mathbb{P}_{\boldsymbol{v}\sim\boldsymbol{F}}\big[(\max_{i\in[n]} v_i) \geq p\big] = p \cdot \big(1 - F(p)\big), && \forall p \geq 0. \end{aligned}$$

In the regret minimization problem, an algorithm $\mathcal{A}$ needs to play a $T$-round repeated game, as follows:

- Initially, $\mathcal{A}$ has no information of the value distributions $\boldsymbol{F}$ (except for their independence and $[0, 1]$ support).
- Each round $t = 1, 2, \dots, T$ refers to an *independent* trial $\boldsymbol{v}^t \sim \boldsymbol{F}$ of the Uniform Pricing mechanism:
  $\mathcal{A}$ posts a price $p^t$, acquires whether the sale succeeds or not $z^t = \mathbb{1}[(\max_{i\in[n]} v_i^t) \geq p^t] \in \{0, 1\}$, and thus accumulates an amount of $p^t \cdot z^t$ revenue.

The regret minimization problem asks for the *minimax regret* $\mathrm{Regret}(T) \in [0, T]$ accumulated throughout the game, against the optimal Uniform Pricing revenue $R(p^{\mathrm{opt}})$:

$$\mathrm{Regret}(T) \ :=\ \min_{\mathcal{A}} \max_{\boldsymbol{F}} \mathbb{E}_{\boldsymbol{F},\mathcal{A}}\Big[\textstyle\sum_{t=1}^T \big(R(p^{\mathrm{opt}}) - p^t \cdot z^t\big)\Big].$$

Below in Section B.1, we will give a black-box reduction from *pricing query complexity* lower bounds to *regret* lower bounds. This reduction in combination with Lemmas 11 and 16 (after suitable scales of $\varepsilon \in (0, 1)$) directly gives Corollaries 23 and 24.

**Corollary 23.** *For two (or more) regular distributions, the minimax regret of Uniform Pricing is $\Omega(T^{2/3})$.*

**Corollary 24.** *For three (or more) MHR distributions, the minimax regret of Uniform Pricing is $\Omega(T^{2/3})$.*

### B.1 A Reduction from Identification to Regret Minimization

Given a sufficiently small $\varepsilon \in (0, 1)$ and a sufficiently large $K := K(\varepsilon)$. Suppose that we have one base instance $\boldsymbol{F}^*$ and $K$ hard instances $\{\boldsymbol{F}^i\}_{i\in[K]}$ that, for some disjoint intervals $(s^i, t^i)$ for $i \in [K]$, satisfy the following three conditions:

- $R^*(p)$ is maximized by every price $p \in (s^1, t^K]$.
- $R^*(p) = R^i(p)$ for $p \notin (s^i, t^i)$.
- $\max_{p\in(s^i,t^i]} \big(R^i(p) - R^*(p)\big) \geq \varepsilon$, for each $i \in [K]$.

Note that, after suitable scales of $\varepsilon \in (0, 1)$, these conditions hold for each of the lower-bound constructions in Sections 3 and 4.

**Lemma 25** (Identification Upper Bounds)**.** *Given a universal constant $\alpha \in (0, 1)$, if there exists an $O(T^\alpha)$-regret algorithm $\mathcal{A}$, then there exists an $O(\varepsilon^{-\frac{1}{1-\alpha}})$-query identification algorithm $\mathcal{B}$.*

Proof. Without loss of generality, for some universal constant $c > 0$, the algorithm $\mathcal{A}$ has a regret bound $\mathrm{Regret}(T) \leq c \cdot T^\alpha$; we would run it for $T = \lceil (3c/\varepsilon)^{\frac{1}{1-\alpha}} \rceil = O(\varepsilon^{-\frac{1}{1-\alpha}})$ rounds, and let $T^i$ denote how many pricing queries are made within the index-$i$ modification interval $(s^i, t^i)$, for $i \in [K]$.

Denote by $\mathbb{E}^i[\cdot]$ the expectations in each possibility $i \in [K]$. We can deduce that

$$\tfrac{1}{T} \cdot \mathbb{E}^i[T - T^i] \ \leq\ \tfrac{1}{T} \cdot \varepsilon^{-1} \cdot \mathrm{Regret}(T) \ \leq\ c \cdot \varepsilon^{-1} \cdot T^{\alpha-1} \ \leq\ \tfrac{1}{3}.$$

Here the first step uses $\mathrm{Regret}(T) \leq \varepsilon \cdot \mathbb{E}^i[T - T^i]$ (obvious). The second step uses $\mathrm{Regret}(T) \leq c \cdot T^\alpha$. And the last step uses $T = \lceil (3c/\varepsilon)^{\frac{1}{1-\alpha}} \rceil$.

Thus, the identification algorithm $\mathcal{B}$ can, after the $T = O(\varepsilon^{-\frac{1}{1-\alpha}})$ pricing queries, simply output each $\boldsymbol{F}^i$ for $i \in [K]$ with probability $T^i/T$;[12] the above equation immediately implies that $\mathcal{B}$ successes with probability $\geq \frac{2}{3}$. This finishes the proof. □

Corollary 26 is a direct implication of Lemma 25.

**Corollary 26** (Regret Lower Bounds)**.** *Given a universal constant $\beta = \frac{1}{1-\alpha} > 1$, if any identification algorithm $\mathcal{B}$ has query complexity $\Omega(\varepsilon^{-\beta})$, then any regret minimization algorithm $\mathcal{A}$ has regret $\Omega(T^{1-1/\beta})$.*

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[12] If there are pricing queries made outside all modification intervals $(s^i, t^i)$ for $i \in [K]$, then $\mathcal{B}$ can output arbitrarily with the remaining probability $(1 - \sum_{i\in[K]} T_i/T)$.