Tight Revenue Gaps among Simple Mechanisms

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Abstract

We consider a fundamental problem in microeconomics: Selling a single item among a number of buyers whose values are drawn from known independent and regular distributions. There are four widely-used and widely-studied mechanisms in this literature: Anonymous Posted-Pricing (AP), Second-Price Auction with Anonymous Reserve (AR), Sequential Posted-Pricing (SPM), and Myerson Auction (OPT). Myerson Auction is optimal but complicated, which also suffers a few issues in practice such as fairness; AP is the simplest mechanism, but its revenue is also the lowest among these four; AR and SPM are of intermediate complexity and revenue. We study the revenue gaps among these four mechanisms, which is defined as the largest ratio between revenues from two mechanisms. We establish two tight ratios and one tighter bound:

1. SPM/AP. This ratio studies the power of discrimination in pricing schemes. We obtain the tight ratio of roughly 2.62, closing the previous known bounds $[e/(e-1), e]$.
2. AR/AP. This ratio studies the relative power of auction vs. pricing schemes, when no discrimination is allowed. We get the tight ratio of $\pi^2/6 \approx 1.64$, closing the previous known bounds $[e/(e-1), e]$.
3. OPT/AR. This ratio studies the power of discrimination in auctions. Previously, the revenue gap is known to be in interval $[2, e]$, and the lower-bound of 2 is conjectured to be tight [38, 37, 4]. We disprove this conjecture by obtaining a better lower-bound of 2.15.

1 Introduction

How to maximize the expected revenue of a seller, who aims to sell a single item among a number of buyers, is a central problem in microeconomics. The simplest mechanism is the *Anonymous Posted-Pricing* (AP) mechanism. An anonymous posted-pricing mechanism simply posts a price p for all buyers. The item is sold iff at least one buyer values the item higher than or equal to p. If the seller knows value distributions of the buyers, he can choose a proper price p to maximize his expected revenue among all AP mechanisms. Although quite widely-used, this is not the optimal method to sell a single item. The optimal mechanism is the remarkable *Myerson Auction* [46] (which will be denoted by OPT in the following discussions). Compared to AP mechanism, Myerson Auction is considerably more complicated, mainly due to two reasons:

1. It discriminates different buyers with different value priors. This may incur some fairness issues, and is not feasible in some markets.

2. It is an *auction* rather than a *pricing* scheme, thus involves more communications between the seller and the buyers. This may also raise some privacy concerns for the buyers, since they need to report their private values, rather than make take-it-or-leave-it decisions.

These complications and some other undesirable issues hinder the prevalence of Myerson Auction. To address these issues, two simple mechanisms with intermediate complexity (w.r.t. OPT and AP) are well-studied in the literature, and are widely-used in practice: (1) To avoid discrimination, one may use the *Second-Price Auction with Anonymous Reserve* (AR) [38]; and (2) To reduce communications between the seller and the buyers, one may use the *Sequential Posted-Pricing* (SPM) mechanism [18, 19]. We defer formal definitions of these mechanisms to Section 2.

These four mechanisms form the lattice structure in Figure 1, both in terms of simplicity, and in terms of revenue but in reversed order. It is well-known in microeconomics that there are revenue gaps between any two of them. But how large can these gaps possibly be?

Quantitative analysis of these gaps is also a striking theme in the theory of algorithmic mechanism design. To measure the gaps among mechanisms, the approximation ratio, which is originated from theoretical computer science, turns out to be a very powerful language. There is rich literature that studies revenue gaps/approximation ratios among different mechanisms [13, 33, 9, 34, 41, 21, 22, 32, 26, 4, 25].

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¶This work was completed while the author visited the Institute for Theoretical Computer Science at Shanghai University of Finance and Economics.

1.1 Our Results In the environment with asymmetric regular distributions, no tight revenue gaps were previously known, for any pair of these four basic mechanisms. In this paper, we establish two tight ratios and one tighter bound, where the last one between OPT and AR disproves a conjecture of near a decade [38, 37, 4].

SPM vs. AP. This comparison studies the power of discrimination in pricing schemes. We obtain the tight ratio of 2.62 in asymmetric regular setting. In the other three settings, namely asymmetric general, i.i.d. general and i.i.d. regular settings, tight ratios were known to be n [4], 2 and $e/(e-1) \approx 1.58$ [37, 26], respectively. The last two ratios can be got by combining results in [46, 43, 39], which was first observed in [35].

AR vs. AP. This comparison studies the relative power of auction vs. pricing schemes, when no discrimination is allowed. We get the tight ratio of $\pi^2/6 \approx 1.64$ in the most general setting. This ratio is also tight, in both of asymmetric regular and i.i.d. general[1] settings. In the most special i.i.d. regular setting, where AR is exactly OPT, an upper-bound of $e/(e-1) \approx 1.58$ was obtained in [19], and was shown to be tight afterwards [37].

OPT vs. AR. This comparison studies the power of discrimination in auctions. Up till now, tight ratios are got in all settings [46, 37, 4], except for the one with asymmetric regular distributions. The problem in this setting was initiated by Hartline and Roughgarden [38], who proposed the so-called *simple versus optimal paradigm*. In that paper, they proved an upper-bound of 4 (which was improved to e subsequently [4]), and provided a 2-approximate lower-bound example. The ratio of 2 was conjectured to be tight in that paper, and had been remaining to be the known worst-case instance in the last decade. Nevertheless, in Section 5 we disprove this conjecture by proposing a sharper 2.15-approximate lower-bound example.

These three bounds also improve some other bounds by implication. For example, by Alaei et al. [4], the tight ratio between OPT and AP was known to be in interval $[2.23, e]$. Given our tight ratio of 2.62 for SPM vs. AP, this interval is narrowed to be $[2.62, e]$.

For the two tight ratios, worst cases are reached when there are infinitely many buyers. Either upper-bound is proved by writing the revenue gap as the objective of a mathematical program. This approach was initiated in [21, 4]. Recently, a similar approach was also used to give a tight *Price of Anarchy* for multi-unit auction [11]. Our work further illustrates that this is a powerful approach to get tight ratios. En route, we develop a number of tools to handle these mathematical programs, which may find their applications in future work.

Previously, most known tight revenue gaps are achieved with "small" worst-case instances [13, 38, 19, 40], say two buyers for example. For the gap between OPT and AR, it was conjectured in [38] that a two-buyer example was tight. Our tighter three-buyer and four-buyer lower-bound examples suggest the worst case may be achieved by an infinite-buyer instance, and mathematical program might serve as the right approach towards the ultimate solution.

1.2 Open Problems and Conjectures Although we get two tight ratios, there are three ones left open, among these four basic mechanisms, namely OPT vs. AP, AR and SPM respectively (in the lattice structure, AR and SPM are incomparable, thus the ratio between them is not that interesting). The main obstacle to getting these tight ratios by current approaches is that we do not have a good method to express OPT in a mathematical program. It is easy to write AP, and in this paper we develop tools to deal with AR and SPM, which enable the proof of our results.

By our work, the ratio of OPT to AP is now in a very narrow interval $[2.62, e]$. We conjecture that the lower-bound of 2.62 is tight, due to the following two reasons. First, in our tight example for SPM vs. AP (see Example 2 in Appendix A.4), OPT *does* achieve the same revenue as SPM (suggested by Lemma 3.2 in Section 3, and Remark 3 in Appendix A.1). Second, in all other three settings, namely asymmetric general, i.i.d. general and i.i.d. regular settings, AP has the same revenue gaps w.r.t. OPT and SPM (see Table 2 and Table 3 in Section 6).

By our improved lower-bound, the tight ratio for OPT vs. AR is now in interval $[2.15, e]$. We believe that neither the lower-bound nor the upper-bound is tight. On the other hand, by Correa et al. [25, 24], the ratio of OPT to SPM is in interval $[1.34, 1.50]$. We slightly believe the lower-bound is tight. For both problems, tools developed here for AR and SPM may be helpful, and the tight ratios may be achieved by infinite-buyer instances, similar to our tight examples.

1.3 Further Related Works This line of work was initiated by Hartline and Roughgarden [38], who showed that the revenue gap between OPT and AR is in interval $[2, 4]$. The most related work of Alaei et

[1]Notably, in i.i.d. settings (regular or general), this result completes the last piece of the puzzle (see Table 2).

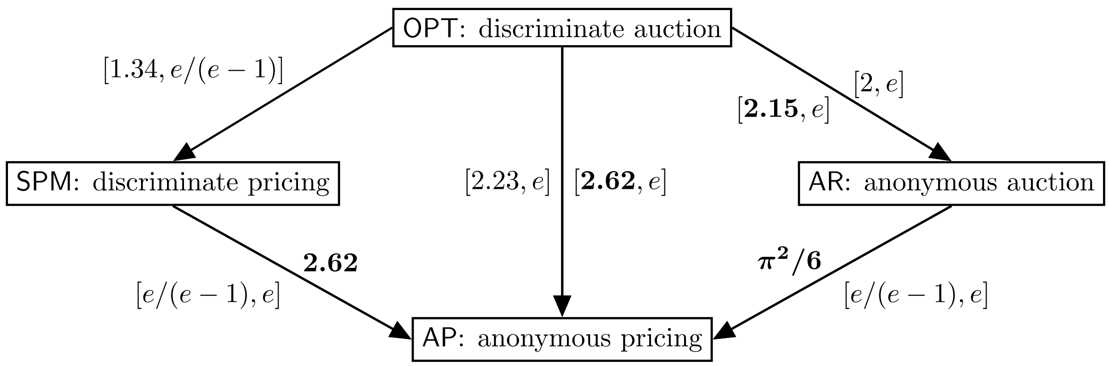


Figure 1: Revenue gaps among basic mechanisms in asymmetric regular setting; our results are marked in bold. A thorough summary of results in all settings can be referred in Section 6.

al. [4] initiated the mathematical program approach in this context, and gave an improved upper-bound of e for OPT vs. AP. They exploited a technique called *Ex-Ante Relaxation*, in that it is difficult to directly quantify the revenue from OPT. Ex-ante relaxation is a "fake" mechanism. Nevertheless, it gives an upper-bound for OPT, and is easy to deal with in mathematical programs. The upper-bound of e is the tight ratio between that and AP. Chawla et al. [19] introduced the notion of ex-ante relaxation, which is originated from SPM, and was refined by Alaei [2] later.

Chawla et al. [18, 19] initiated the study for OPT vs. SPM, by acquiring an upper-bound of $e/(e-1) \approx 1.58$ in asymmetric settings (regular or general). Later, Yan [51] showed that the same ratio holds in more general settings. Very recently, this $e/(e-1)$ barrier was beaten in [6, 10, 24]. Moreover, Correa et al. [25] obtained the tight ratio of 1.34 in i.i.d. settings (regular or general). Hajiaghayi et al. [35] first found the connection between this problem and the notion of prophet inequalities in optimal stopping theory [42, 43]. The last decade has seen extensive progress on (single and combinatorial) prophet inequalities [7, 40, 48, 49, 1, 30, 29] (see the survey by Lucier [45] and the references therein for more literature), due to their appealing applications in algorithm and mechanism designs.

In multi-item environments, optimal mechanisms could be even more complicated. There is a long line of works on proving that simple mechanisms constantly approximate the optimal [18, 12, 19, 36, 3, 44, 8, 14, 31, 50, 52, 15, 20, 5, 16, 28, 27]. All these works assume that the distributions of different buyers are independent. To sell a single item among correlated buyers, Myerson Auction may not be optimal. For this, some simple and approximately optimal mechanisms were proposed, such as look-ahead auction and k-look-ahead auction [47, 23]. For a full survey on simple auctions, one can refer to the book entitled "Mechanism Design and Approximation" by Hartline [37].

2 Preliminaries

Throughout the paper, we consider the single-item environment, where there are n buyers with bids[2] $\{b_i\}_{i=1}^n$ drawn independently from distributions $\{F_i\}_{i=1}^n$. Most of our results are established under the standard assumption that all distributions are regular. Moreover, the family of triangular distributions is widely used in our analysis. We introduce these two concepts below.

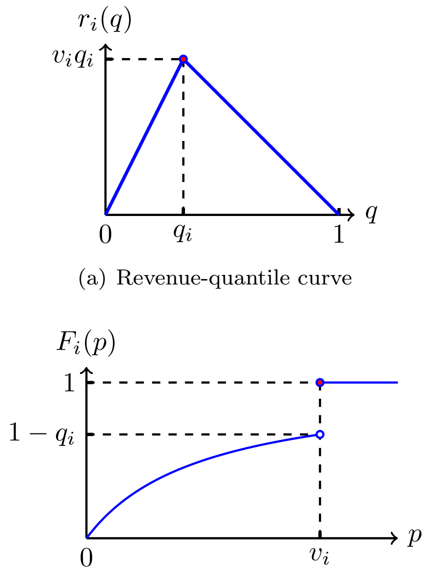
Regular Distributions and Revenue-Quantile Curves. Given a cumulative density function (CDF) $F(p)$ and its probability density function $f(p)$ (which is assumed to be existing), the virtual value function is defined as $\varphi(p) \stackrel{\text{def}}{=} p - \frac{1-F(p)}{f(p)}$, and the revenue-quantile curve is defined as $r(q) \stackrel{\text{def}}{=} q \cdot F^{-1}(1-q)$. By standard notion, distribution $F(p)$ is regular (i.e. $F \in \text{Reg}$) iff its virtual value $\varphi(p)$ is non-decreasing, or equivalently, iff its revenue-quantile curve $r(q)$ is concave. We interchange the definitions whenever one is more convenient for our use.

Triangular Distributions. This family of distributions is first introduced in [4], named according to the shapes of their revenue-quantile curves (as Figure 2 suggests). Parameterized by $v_i \in (0, \infty)$ and $q_i \in (0, 1]$, a triangular distribution $\text{Tri}(v_i, q_i)$'s CDF is defined as:

$$F_i(p) = \begin{cases} \frac{p(1-q_i)}{p(1-q_i)+v_i q_i} & p \in [0, v_i) \\ 1 & p \in [v_i, \infty) \end{cases}.$$

[2] All mechanisms studied in this paper are truthful, so buyers' bids always equal to their true values.

For the triangular distribution $\text{Tri}(N, \frac{1}{N})$, its CDF converges to $F(p) = \frac{p}{p+1}$ when N goes to infinity. We denote this special triangular distribution by $\text{Tri}(\infty)$.



(a) Revenue-quantile curve

(b) CDF

Figure 2: Demonstration of triangular distributions

In this paper, we study the revenue gaps among the following four families of mechanisms.

Anonymous Posted-Pricing (AP). An anonymous posted-pricing mechanism simply posts a price p to all buyers. The item is sold iff at least one buyer values the item no less than p. We use $\mathsf{AP}(p, \{F_i\}_{i=1}^n) \stackrel{\text{def}}{=} p \cdot \left(1 - \prod_{i=1}^n F_i(p)\right)$ to denote the revenue by posting an anonymous price at p, and abuse $\mathsf{AP}(\{F_i\}_{i=1}^n) \stackrel{\text{def}}{=} \max_{p \in [0,\infty)} \{\mathsf{AP}(p, \{F_i\}_{i=1}^n)\}$ to denote the optimal revenue among this family of mechanisms. We often drop the term $\{F_i\}_{i=1}^n$ for notational simplicity (the same below for SPM, AR and OPT), when there is no ambiguity from the context.

Sequential Posted-Pricing (SPM). Given an order $\sigma : [n] \to [n]$ over buyers, the seller sequentially posts price $p_{\sigma^{-1}(i)}$ to the i-th coming buyer. The item is sold to the i-th buyer who first values it $b_{\sigma^{-1}(i)} \geq p_{\sigma^{-1}(i)}$. Given an instance $\{F_i\}_{i=1}^n$, let $\mathsf{SPM}(\sigma, \{p_i\}_{i=1}^n, \{F_i\}_{i=1}^n)$ denote the revenue achieved by a specific pair of σ and $\{p_i\}_{i=1}^n$, and let $\mathsf{SPM}(\sigma, \{F_i\}_{i=1}^n) \stackrel{\text{def}}{=} \max_{\{p_i\}_{i=1}^n} \{\mathsf{SPM}(\sigma, \{p_i\}_{i=1}^n, \{F_i\}_{i=1}^n)\}$ be the revenue from the optimal pricing strategy under order σ. Unless otherwise stated, we assume the seller can choose the order σ, and abuse $\mathsf{SPM}(\{F_i\}_{i=1}^n) \stackrel{\text{def}}{=} \max_{\sigma \in \Pi} \{\mathsf{SPM}(\sigma, \{F_i\}_{i=1}^n)\}$.

Second-Price Auction with Anonymous Reserve (AR). An anonymous reserve mechanism with reserve price p runs in the following way: If there is no buyer who bids above p, the item remains unsold; if there is exactly one buyer biding above p, then sell this item to this buyer with price p; otherwise, the buyer with highest bid gets the item, and pays the second highest bid (which is the remarkable *Second-Price Auction*). Given distributions $\{F_i\}_{i=1}^n$, let $\mathsf{AR}(p, \{F_i\}_{i=1}^n)$ be the revenue by running anonymous reserve mechanism with reserve price p, and let $\mathsf{AR}(\{F_i\}_{i=1}^n) \stackrel{\text{def}}{=} \max_{p \in [0,\infty)} \{\mathsf{AR}(p, \{F_i\}_{i=1}^n)\}$. It is easy to see $\mathsf{AR}(p, \{F_i\}_{i=1}^n) \geq \mathsf{AP}(p, \{F_i\}_{i=1}^n)$, for all $p \in [0, \infty)$, and further $\mathsf{AR}(\{F_i\}_{i=1}^n) \geq \mathsf{AP}(\{F_i\}_{i=1}^n)$. We defer the explicit formula for $\mathsf{AR}(p, \{F_i\}_{i=1}^n)$ to Section 4.

Myerson Auction (OPT). Given distributions $\{F_i\}_{i=1}^n$, Myerson Auction runs in the following way: Each buyer i is associated with a virtual value function $\varphi_i(b_i) \stackrel{\text{def}}{=} b_i - \frac{1-F_i(b_i)}{f_i(b_i)}$. Upon receiving bids, the seller sells the item to the buyer with the highest virtual value (required to be above 0), and charges him a critical price p that is the minimum bid for him to win.

3 Sequential Posted-Pricing vs. Anonymous Posted-Pricing

In this section, we study the revenue gap between SPM and AP, assuming that buyers' values are drawn from regular and independent (not necessarily identical) distributions. This problem is formed in the following program (we safely drop the constraint on $p \in [0, 1]$ as it always holds):

(P1)

$$\max_{\{F_i\}_{i=1}^n \subset \text{Reg}} \mathsf{SPM} = \max_{\sigma \in \Pi, \{p_i\}_{i=1}^n} \{\mathsf{SPM}(\sigma, \{p_i\}_{i=1}^n)\}$$

$$\text{(3.1)} \qquad \text{s.t. } \mathsf{AP}(p) = p \cdot \left(1 - \prod_{i=1}^n F_i(p)\right) \leq 1, \forall p \in (1, \infty)$$

By solving this program optimally, we prove the following theorem.

Theorem 3.1. *For asymmetric regular distributions, the supremum of the ratio of SPM to AP is*

$$1 + \int_1^\infty x \cdot (-\mathcal{Q}'(x)) \cdot e^{-\mathcal{Q}(x)} dx \approx 2.6202,$$

where $\mathcal{Q}(p) \stackrel{def}{=} \ln\left(\frac{p^2}{p^2-1}\right) - \frac{1}{2}Li_2\left(\frac{1}{p^2}\right)$ *and* $Li_2(z) \stackrel{def}{=} \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ *is the polylogarithm function of order* 2.

Proof Overview. Our first step is to show that in the worst case, an instance $\{F_i\}_{i=1}^n$ falls into the family of *triangular distributions*. Afterwards, we clarify it is safe to assume that there is a buyer with distribution $\text{Tri}(\infty)$, who contributes a revenue of 1 to SPM. These are formulated in Section 3.1, after which the annoying objective of Program (P1), $\mathsf{SPM} = \max_{\sigma \in \Pi, \{p_i\}_{i=1}^n} \{\mathsf{SPM}(\sigma, \{p_i\}_{i=1}^n)\}$, can be expressed explicitly.

We further prove in Section 3.2 that the revenue from the remaining buyers is upper-bounded by that from infinitely many "small buyers" (in terms of the possibility of buying the item), which is captured by the integration term in the theorem.

We provide a matching lower-bound example in Appendix A.4. The idea is simple: Use finitely many "small buyers" with triangular distributions to approach the instance in the upper-bound proof. With sufficiently many buyers, SPM can be arbitrarily close to the upper-bound.

3.1 Upper-Bound Analysis I: Reductions

Exploiting the idea in [4], we exhibit a method to tailor a regular instance $\{F_i\}_{i=1}^n$ into a triangular one so that SPM remains the same, while AP tends to decrease.

LEMMA 3.1. *For any regular instance* $\{F_i\}_{i=1}^n$*, there exists a triangular instance* $\{Tri(v_i, q_i)\}_{i=1}^n$ *such that (1)* $\mathsf{SPM}(\{Tri(v_i, q_i)\}_{i=1}^n) \geq \mathsf{SPM}(\{F_i\}_{i=1}^n)$*; and (2)* $\mathsf{AP}(\{Tri(v_i, q_i)\}_{i=1}^n) \leq \mathsf{AP}(\{F_i\}_{i=1}^n)$.

We defer the proof of Lemma 3.1 to Appendix A.1. Transparently, to study the worst case of Program (P1), it suffices to focus on triangular instances. For notational simplicity, we re-index all $\text{Tri}(v_i, q_i)$'s such that $v_1 \geq v_2 \geq \cdots \geq v_n$, and keep using F_i to denote the CDF of $\text{Tri}(v_i, q_i)$. Recall the formula for each F_i defined in Section 2, we rewrite constraint (3.1) as

$$\begin{aligned} \mathsf{AP}(p) &= p \cdot \left(1 - \prod_{i=1}^n F_i(p)\right) \\ &= p \cdot \left\{1 - \prod_{i: v_i \geq p} \left[\frac{p(1-q_i)}{p(1-q_i) + v_i q_i}\right]\right\} \\ &\leq 1 \qquad \forall p \in (1, \infty). \end{aligned} \tag{3.2}$$

In Appendix A.1, we prove the following structural lemma of SPM and OPT.

LEMMA 3.2. *Given a triangular instance* $\{Tri(v_i, q_i)\}_{i=1}^n$ *with* $v_1 \geq v_2 \geq \cdots \geq v_n$*, (1) the optimal sequential posted-price mechanism posts price* $p_i = v_i$ *to the* i*-th buyer, for* i *from* 1 *to* n*; and (2)* $\mathsf{SPM} = \mathsf{OPT} = \sum_{i=1}^n v_i q_i \cdot \prod_{j=1}^{i-1}(1 - q_j)$.

Rearranging constraint (3.2), we rewrite Program (P1) as follows:

(P2)

$$\begin{aligned} \max_{\{\text{Tri}(v_i,q_i)\}_{i=1}^n} \quad & \mathsf{SPM} = \sum_{i=1}^n v_i q_i \cdot \prod_{j=1}^{i-1}(1 - q_j) \\ \text{s.t.} \quad & \sum_{i: v_i \geq p} \ln\left(1 + \frac{v_i q_i}{1 - q_i} \cdot \frac{1}{p}\right) \\ & \leq \ln\left(\frac{p}{p-1}\right) \quad \forall p \in (1, \infty) \\ & v_1 \geq v_2 \geq \cdots \geq v_n \geq 1 \end{aligned} \tag{3.3}$$

We can safely assume that $v_n \geq 1$. Since constraint (3.3) is irrelevant to all v_i's that are smaller than or equal to 1, increasing all these v_i's to 1 directly leads to an improved objective value.

The next two facts further narrow the space of the worst-case instances:

1. Whenever there are two buyers/distributions with the same v_i, we can substitute the buyers with a single (feasible) buyer so that the objective value remains the same.

2. The worst-case instance contains a buyer with distribution $\text{Tri}(\infty)$, i.e. $F_0(p) = \frac{p}{p+1}$ for all $p \in [0, \infty)$. For notational simplicity, we will not explicitly mention this special buyer later.

These two statements are formalized as Lemma A.1 and Lemma A.2, and then proved in Appendix A.1. In sum, the optimal objective values of Program (P3) and Program (P1) are equal.

(P3)

$$\begin{aligned} \max_{\{\text{Tri}(v_i,q_i)\}_{i=1}^n} \quad & \mathsf{SPM} = 1 + \sum_{i=1}^n v_i \cdot q_i \cdot \prod_{j=1}^{i-1}(1 - q_j) \\ \text{s.t.} \quad & \sum_{i: v_i \geq p} \ln\left(1 + \frac{v_i q_i}{1 - q_i} \cdot \frac{1}{p}\right) \\ & \leq \ln\left(\frac{p^2}{p^2-1}\right) \quad \forall p \in (1, \infty) \\ & v_1 > v_2 > \cdots > v_n \geq 1 \end{aligned} \tag{3.4}$$

3.2 Upper-Bound Analysis II: Optimal Solution We adapt techniques developed by Alaei et al. [4] to deal with Program (P3). It is easy to check that $\frac{1}{p}\ln(1+x) \le \ln\left(1+\frac{x}{p}\right)$ for all $p \ge 1$ and $x \ge 0$. We apply this inequality to constraint (3.4), and drop all constraints other than $p \in \{v_1, v_2, \cdots, v_n\}$, resulting in

$$(3.5) \qquad \sum_{i=1}^{k} \ln\left(1+\frac{v_i q_i}{1-q_i}\right) \le \mathcal{R}(v_k) \qquad \forall k \in [n],$$

where $\mathcal{R}(p) \overset{\text{def}}{=} p\ln\left(\frac{p^2}{p^2-1}\right)$. Recall that $\mathcal{Q}(p) = \ln\left(\frac{p^2}{p^2-1}\right) - \frac{1}{2}Li_2\left(\frac{1}{p^2}\right)$, the following lemma (which is proved in Appendix A.2) suggests that these two functions are inherently correlated.

LEMMA 3.3. *$\mathcal{R}'(p) = p\cdot\mathcal{Q}'(p)$ and $\mathcal{R}'(p) < \mathcal{Q}'(p) < 0$ when $p \in (1,\infty)$, and*

$$(3.6) \qquad \begin{aligned} \lim_{p\to\infty} \mathcal{R}(p) = \lim_{p\to\infty} \mathcal{Q}(p) = 0 \\ \lim_{p\to 1^+} \mathcal{R}(p) = \lim_{p\to 1^+} \mathcal{Q}(p) = \infty. \end{aligned}$$

After the aforementioned relaxation, optimum of the new program turns out to be reached when constraint (3.5) is tight for each $k \in [n]$. This is formalized as the following technical lemma.

LEMMA 3.4. *Given a triangular instance $\{Tri(v_i, q_i)\}_{i=1}^n$ that constraint (3.5) is not tight for some $k \in [n]$, there exists a triangular instance $\{Tri(\overline{v}_i, \overline{q}_i)\}_{i=1}^n$ such that (1) $\sum_{i=1}^{k} \ln\left(1+\frac{\overline{v}_i\overline{q}_i}{1-\overline{q}_i}\right) = \mathcal{R}(\overline{v}_k)$, for all $k \in [n]$; and (2) $\sum_{i=1}^{n} \overline{v}_i\overline{q}_i \cdot \prod_{j=1}^{i-1}(1-\overline{q}_j) \ge \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j)$.*

Denote $v_0 \overset{\text{def}}{=} \infty$ for notational simplicity. Given the tightness of constraint (3.5), we acquire the recursive formula for each q_k that $\ln\left(1+\frac{v_k q_k}{1-q_k}\right) = \mathcal{R}(v_k) - \mathcal{R}(v_{k-1})$, where we apply Lemma 3.3 that $\mathcal{R}(v_0) = 0$. After being rearranged, it is equivalent to

$$(3.7) \qquad q_k = \frac{e^{\mathcal{R}(v_k)-\mathcal{R}(v_{k-1})}-1}{v_k + e^{\mathcal{R}(v_k)-\mathcal{R}(v_{k-1})}-1} \qquad \forall k \in [n].$$

Equipped with these formulas, we prove the following mathematical facts in Appendix A.2.

LEMMA 3.5. *For each $i \in [n]$,*

$$(3.8) \quad \begin{aligned} & v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j) - \int_{v_i}^{v_{i-1}} x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx \\ \le & \left[\prod_{j=1}^{i-1}(1-q_j) - \prod_{j=1}^{i}(1-q_j)\right] - \left(e^{-\mathcal{Q}(v_{i-1})} - e^{-\mathcal{Q}(v_i)}\right). \end{aligned}$$

LEMMA 3.6. *$1 \le p^* \le v_n$, where $p^* \overset{def}{=} \mathcal{Q}^{-1}\left(-\sum_{i=1}^{n}\ln(1-q_i)\right)$.*

With the p^* defined in Lemma 3.6, we are ready to complete the upper-bound part of Theorem 3.1. By Lemma 3.3, $e^{-\mathcal{Q}(v_0)} = 1$, and $\mathcal{Q}'(p) \le 0$ for all $p \in (1,\infty)$. Applying Lemma 3.5 over all $i \in [n]$,

$$\begin{aligned} & \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j) - \int_{v_n}^{\infty} x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx \\ \le & 1 - \prod_{j=1}^{n}(1-q_j) - \left(e^{-\mathcal{Q}(v_0)} - e^{-\mathcal{Q}(v_n)}\right) \\ = & e^{-\mathcal{Q}(v_n)} - e^{-\mathcal{Q}(p^*)} \le \int_{p^*}^{v_n} (-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx \\ \le & \int_{p^*}^{v_n} x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx. \end{aligned}$$

Together with Lemma 3.6 that $p^* \ge 1$, we can conclude that

$$\begin{aligned} & 1 + \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j) \\ \le & 1 + \int_{p^*}^{\infty} x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx \\ \le & 1 + \int_{1}^{\infty} x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx. \end{aligned}$$

REMARK 1. *In the above context, $\{Tri(v_i, q_i)\}_{i=1}^n$ is replaced by $\{Tri(\overline{v}_i, \overline{q}_i)\}_{i=1}^{\infty}$, that is, a spectrum of "small" (all $\overline{q}_i$'s tend to 0^+) triangular distributions lying in interval $[p^*, \infty)$. For all $p \ge p^*$,*

1. *The total quantity of $\overline{q}_i$'s cumulating in interval $[p,\infty)$ is given by $\mathcal{Q}(p)$;*
2. *$\prod_{i:\overline{v}_i \ge p}(1-\overline{q}_i)$ approaches to $e^{-\mathcal{Q}(p)}$, in that all $\overline{q}_j$'s go to 0^+.*

As per these, it is easy to see SPM$(\{Tri(\overline{v}_i, \overline{q}_i)\}_{i=1}^{\infty}) = 1 + \int_{p^*}^{\infty} x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx$, which is ensured (by Lemma 3.5 essentially) to be greater than or equal to SPM$(\{Tri(v_i, q_i)\}_{i=1}^{n})$.

4 Anonymous Reserve vs. Anonymous Posted-Pricing

In this section, we study the revenue gap between AR and AP. We first obtain the CDF's of the highest and the second highest bids from $\{F_i\}_{i=1}^n$, which are respectively denoted by $D_1(p)$ and $D_2(p)$. For $D_1(p)$, we have

$$\begin{aligned}D_1(p) &= \Pr\left\{\bigcap_{i=1}^n [b_i \le p]\right\}\\ &= \prod_{i=1}^n \Pr\{b_i \le p\} = \prod_{i=1}^n F_i(p).\end{aligned}$$

For $D_2(p)$, the event that the second highest bid is no more than p can be partitioned into the following $(n+1)$ disjoint sub-events: (A_0) the highest bid is no more than p; (A_i for each $i \in [n]$) bid b_i is larger than p, while all other bids are no more than p. Therefore, we have

$$\begin{aligned}D_2(p) &= \Pr\{A_0\} + \sum_{i=1}^n \Pr\{A_i\}\\ &= \prod_{i=1}^n F_i(p) + \sum_{i=1}^n (1 - F_i(p)) \cdot \prod_{j \ne i} F_j(p)\\ &= D_1(p) \cdot \left[1 + \sum_{i=1}^n \left(\frac{1}{F_i(p)} - 1\right)\right].\end{aligned}$$

With these notations, it is easy to see that $\mathsf{AP}(p) = p \cdot (1 - D_1(p))$. Additionally, the following lemma establishes an explicit formula for $\mathsf{AR}(p)$, by using $D_1(p)$ and $D_2(p)$.

Lemma 4.1. ([17]) *For any reserve price $p \in [0, \infty)$,*

$$\mathsf{AR}(p) = p \cdot (1 - D_1(p)) + \int_p^\infty (1 - D_2(x))\, dx.$$

This formula was first introduced in [17], and plays an important role in the proof of our tight ratio of AR to AP. It is also used in the next section to get a better lower-bound between OPT and AR. For the sake of completeness, we provide a proof in Appendix B.1.

Given these formulas for AR and AP, the revenue gap between AR and AP can be characterized by the following program.

(P4)

$$\max_{\{F_i\}_{i=1}^n} \max_{p \in [0,\infty)} \left\{p \cdot (1 - D_1(p)) + \int_p^\infty (1 - D_2(x))\, dx\right\}$$

(4.9)
$$\text{s.t.} \mathsf{AP}(x) = x \cdot [1 - D_1(x)] \le 1, \quad \forall x \in [0, \infty)$$

We prove the upper-bound in (the most general) asymmetric general setting, which automatically gives an upper-bound to the rest two settings. In Appendix B.2 and Appendix B.3, we respectively construct a matching lower-bound example with i.i.d. general and asymmetric regular distributions, which implies the tightness in all three settings.

Theorem 4.1. *The supremum of the ratio of AR to AP is $\frac{\pi^2}{6} \approx 1.6449$. The same tight ratio holds in all the three settings: asymmetric general, asymmetric regular and i.i.d. general settings.*

Remark 2. *Intuitively, we obtain Theorem 4.1 as follows. Recall Program P1 in Section 3, the worst case is achieved when constraint (3.1) is tight, for all $p \in (1, \infty)$. We simply "guess" the worst case of Program (P4) can be captured in the same circumstance, which turns out to be the right way.*

Proof. [Proof of Theorem4.1 (Upper-Bound Part)] Define

$$\Phi_1(p) \overset{\text{def}}{=} \begin{cases} 0 & p \in [0, 1] \\ 1 - \frac{1}{p} & p \in (1, \infty) \end{cases}.$$

One can easily see from constraint (4.9) that $\Phi_1(p)$ stochastically dominates $D_1(p)$, the distribution of the highest bid. That is, $D_1(p) \ge \Phi_1(p)$ for all $p \in [0, \infty)$. Furthermore,

$$\begin{aligned}D_2(p) &= D_1(p) \cdot \left[1 + \sum_{i=1}^n \left(\frac{1}{F_i(p)} - 1\right)\right]\\ &\ge D_1(p) \cdot \left[1 + \sum_{i=1}^n \ln\left(\frac{1}{F_i(p)}\right)\right]\\ &= D_1(p) \cdot [1 - \ln D_1(p)],\end{aligned}$$

where the inequality follows from the fact that $x \ge \ln(1 + x)$ for all $x \ge 0$.

Define function $d(x) \overset{\text{def}}{=} x(1 - \ln x)$, it is easy to check that $d(x)$ is increasing when $x \in (0, 1]$, and that $\lim_{x \to 0^+} d(x) = 0$. As per these, we know $D_2(p) \ge d(\Phi_1(p)) = \Phi_2(p)$ for all $p \in [0, \infty)$, where

$$\Phi_2(p) \overset{\text{def}}{=} \begin{cases} 0 & p \in [0, 1] \\ \left(1 - \frac{1}{p}\right) \cdot \left[1 - \ln\left(1 - \frac{1}{p}\right)\right] & p \in (1, \infty) \end{cases}.$$

Accordingly, the optimal objective value of Program (P4) is bounded from above by

$$\max_{p \in [0,\infty)} \left\{p \cdot (1 - \Phi_1(p)) + \int_p^\infty (1 - \Phi_2(x))\, dx\right\}.$$

1. For all $p \in [0, 1]$, $p \cdot (1 - \Phi_1(p)) + \int_p^\infty (1 - \Phi_2(x))\, dx = 1 + \int_1^\infty (1 - \Phi_2(x))\, dx$;

2. For all $p \in (1,\infty)$, $p \cdot (1-\Phi_1(p)) + \int_p^\infty (1-\Phi_2(x))\,dx < 1 + \int_1^\infty (1-\Phi_2(x))\,dx$.

Putting everything together, the optimal objective value of Program (P4) is no more than

$$
\begin{aligned}
&1+\int_1^\infty (1-\Phi_2(x))\,dx \\
&\overset{(\ddagger)}{=} 1+\int_1^\infty \left[\frac{1}{x}-\left(1-\frac{1}{x}\right)\cdot\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{1}{x^k}\right]dx \\
&= 1+\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\cdot\int_1^\infty \frac{1}{x^{k+1}}dx \\
&= 1+\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)} = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6},
\end{aligned}
\tag{4.10}
$$

where (‡) follows from Taylor series.

5 Myerson Auction vs. Anonymous Reserve

In asymmetric regular setting, the tight ratio of OPT to AR was conjectured to be 2 in [38]. In that paper, a two-buyer instance (Example 1) was proposed, and was conjectured to be the worst case. Nevertheless, here we disprove this conjecture by constructing sharper instances.

EXAMPLE 1. ([38]) *Suppose there are two buyers: The first buyer's bid is drawn from the so-called "equal-revenue" distribution $F(p) = 1-\frac{1}{p}$, and the second buyer has a deterministic bid of* 1*. While* $AR(p) = AP(p) = 1$ *for all* $p \in [1,\infty)$*,* $OPT = SPM = 2$ *(e.g. by sequentially posting price* ∞ *to the first buyer, and posting price* 1 *to the second buyer).*

The following lemma is confirmed in Appendix C, and would be useful for proving Theorem 5.1.

LEMMA 5.1. *Given a triangular instance* $\{Tri(v_i,q_i)\}_{i=1}^n$ *with* $v_1 \ge v_2 \cdots \ge v_n > v_{n+1} \overset{def}{=} 0$*, the maximum of* $AR(p)$ *is achieved by* $p = v_i$ *for some* $i \in [n]$.

THEOREM 5.1. *For the ratio of OPT to AR, there exist a three-buyer triangular instance* $\{Tri(v_i,q_i)\}_{i=0}^2$ *with ratio of* 2.1361*, and a four-buyer triangular instance* $\{Tri(\overline{v}_i,\overline{q}_i)\}_{i=0}^3$ *with ratio of* 2.1596.

Proof. Our four-buyer triangular instance $\{\text{Tri}(\overline{v}_i,\overline{q}_i)\}_{i=1}^n$ is given in the following Table 5. Based on Lemma 5.1 in the above, numerical calculation shows that (1) $\mathsf{AR}(\overline{v}_i) \approx 1.0000$ for all $i \in \{0,1,2,3\}$, therefore $\mathsf{AP} \approx 1.0000$; and that (2) $\mathsf{OPT} = 1 + \sum_{i=1}^{3} \overline{v}_i\overline{q}_i \cdot \prod_{j=1}^{i-1}(1-\overline{q}_j) \approx 2.1596$.

To convey the underlying idea, we would construct the three-buyer instance explicitly.

1. Let $\text{Tri}(v_0,q_0)$ be $\text{Tri}(\infty)$, i.e. $F_0(p) = \frac{p}{p+1}$ in terms of CDF. It is easy to see that for all $p \in (v_1,\infty)$, $\mathsf{AR}(p) = p\cdot(1-F_0(p)) = \frac{p}{p+1} \le 1$. Specifically, $\mathsf{AR}(v_0) = 1$.
2. Let $v_1 \ge 1$ and $q_1 = \frac{1}{v_1^2}$. We have $\mathsf{AR}(v_1) = \mathsf{AP}(v_1) = v_1 \cdot \left[1 - \frac{v_1}{v_1+1}\cdot\frac{(1-q_1)\cdot v_1}{(1-q_1)\cdot v_1 + v_1 q_1}\right] = 1$, since the integration term involved in the formula for $\mathsf{AR}(p)$ equals to 0 whenever $p \ge v_1$.
3. Let v_2 be the root of $v_2 + \frac{v_1}{1+v_1-v_1^2}\cdot \ln\left[\frac{1+v_1}{1+v_2}\cdot\frac{v_2(v_1^2-1)+v_1}{v_1^3}\right] = 1$ and $q_2 = 1$. Basically, this is a buyer with a deterministic bid of v_2 (which means that $\mathsf{AP}(v_2) = v_2$). Reuse $D_2(p)$ to denote the CDF of second highest bid. For all $x \in (v_2, v_1]$,
$$
\begin{aligned}
1-D_2(x) &= \big(1-F_0(x)\big)\cdot\big(1-F_1(x)\big) \\
&= \frac{v_1 q_1}{(x+1)\cdot[(1-q_1)\cdot x + v_1 q_1]},
\end{aligned}
$$
thus $\mathsf{AR}(v_2) = \mathsf{AP}(v_2) + \int_{v_2}^{v_1}(1-D_2(x))\,dx = v_2 + \frac{v_1}{1+v_1-v_1^2}\cdot \ln\left[\frac{x+1}{(v_1^2-1)\cdot x + v_1}\right]\Bigg|_{v_2}^{v_1} = 1$, where the last equality follows from the definition of v_2.

Based on Lemma 5.1 in the above, $\mathsf{AR} = \max\{\mathsf{AR}(v_0),\mathsf{AR}(v_1),\mathsf{AR}(v_2)\} = 1$. Moreover, we know from Lemma 3.2 that $\mathsf{OPT} = 1 + v_1q_1 + v_2q_2\cdot(1-q_1) = 1 + \frac{1}{v_1} + v_2\cdot\left(1-\frac{1}{v_1^2}\right)$. By choosing $v_1 \approx 1.5699$, numerical calculation suggests that $\mathsf{OPT} \approx 2.1361$ and $v_2 \approx 0.8399$, as Figure 3 illustrates.

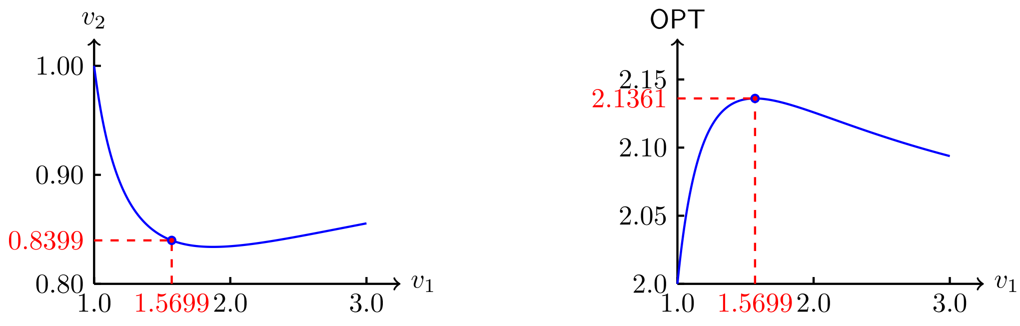
6 Summary of Known Ratios

In this paper, we focus on revenue gaps among OPT, SPM, AR and AP. There is another simple mechanism in the family of sequential posted-pricing that receives lots of attentions, referred as the *Order-Oblivious Posted-Pricing* (OPM) mechanism in [19]. OPM basically characterizes the best pricing strategy and revenue, when buyers come in worst-case (adversarial) order. It is worth studying, since in some practical cases the seller cannot control the order of buyers. The formal definition of OPM is the following:

$$
\mathsf{OPM}(\{F_i\}_{i=1}^n) \overset{\text{def}}{=} \min_{\sigma\in\Pi}\left\{\max_{\{p_i\}_{i=1}^n}\{\mathsf{SPM}(\sigma,\{p_i\}_{i=1}^n,\{F_i\}_{i=1}^n)\}\right\}.
$$

$\text{Tri}(\overline{v}_0, \overline{q}_0) = \text{Tri}(\infty)$	$\text{Tri}(\overline{v}_1, \overline{q}_1) = \text{Tri}(1.8512, 0.2918)$
$\text{Tri}(\overline{v}_2, \overline{q}_2) = \text{Tri}(0.9700, 0.6138)$	$\text{Tri}(\overline{v}_3, \overline{q}_3) = \text{Tri}(0.7231, 1.0000)$

Table 1: Four-buyer instance



(a) v_2-v_1 curve (b) OPT-v_1 curve

Figure 3: Demonstration of three-buyer instances with ratios of OPT to AR greater than 2

In Table 2 and Table 3, we conclude current state-of-art results of gaps among these mechanisms, for both i.i.d. and asymmetric distributions, and for both regular and general distributions. In these results, an interval basically gives lower-bound and upper-bound, while a number means this bound is tight. Recall the lattice structure in Figure 1, both of SPM and OPM are incomparable with AR, and SPM and OPM are same for i.i.d. distributions.

It is always an interesting subject to study gaps among mechanisms. Since some bounds in Table 3 are still not tight, an obvious open question here is to close these gaps, which would give us better understandings of distinctions and relative powers of these mechanisms.

Acknowledgement

We are grateful to Qi Qi and Simai He for many helpful discussions and suggestions.

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	I.I.D. Regular	Reference	I.I.D. General	Reference
OPT/SPM	1.3415	[25]	1.3415	[25]
OPT/AR	1	[46]	2	[19, 37, 26, 46, 43, 39, 35]
OPT/AP	$e/(e-1)$	[19, 26, 46, 43, 39, 35]		
SPM/AP				
AR/AP			$\pi^2/6$	Theorem 4.1

Table 2: Summarized results for i.i.d. distributions

	Asym. Regular	Reference	Asym. General	Reference
OPT/SPM	$[1.3415, e/(e-1)]$	[51, 25]	$[1.3415, e/(e-1)]$	[51, 25]
OPT/OPM	2	Example 1, [42, 43]	2	Example 1, [42, 43]
SPM/OPM				
OPT/AR	$[2.1596, e]$	Theorem 5.1, [4, 38]	n	[4]
OPT/AP	$[2.6202, e]$	Theorem 3.1, [4]		
SPM/AP	2.6202	Theorem 3.1		
OPM/AP	$[e/(e-1), 2.6202]$	Theorem 3.1, [26]		
AR/AP	$\pi^2/6$	Theorem 4.1	$\pi^2/6$	Theorem 4.1

Table 3: Summarized results for asymmetric distributions

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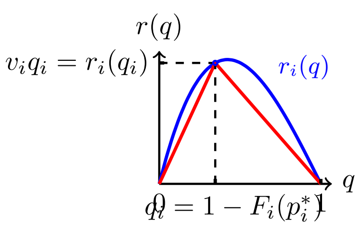
A Missing Proofs in Section 3

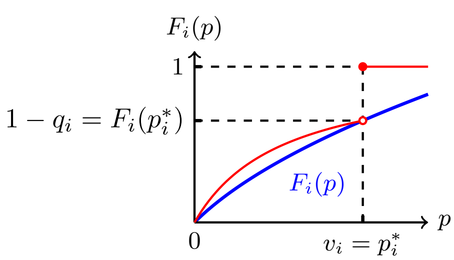
A.1 Missing Proofs in Section 3: Reductions

[Lemma 3.1]. *For any regular instance $\{F_i\}_{i=1}^n$, there exists a triangular instance $\{Tri(v_i, q_i)\}_{i=1}^n$ such that (1) $\mathsf{SPM}(\{Tri(v_i, q_i)\}_{i=1}^n) \geq \mathsf{SPM}(\{F_i\}_{i=1}^n)$; and (2) $\mathsf{AP}(\{Tri(v_i, q_i)\}_{i=1}^n) \leq \mathsf{AP}(\{F_i\}_{i=1}^n)$.*

Proof. Let σ^* and $\{p_i^*\}_{i=1}^n$ respectively be the buyers' coming order and posted prices of the optimal $\mathsf{SPM}(\{F_i\}_{i=1}^n)$. We define $\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^n$ to be the instance that $v_i = p_i^*$ and $q_i = 1 - F_i(p_i^*)$ for all $i \in [n]$, which is illustrated in Figure 4(a). We claim that $\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^n$ satisfies properties above.



(a) Revenue-quantile curves



(b) CDF curves

Figure 4: Transformation from $\{F_i\}_{i=1}^n$ to $\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^n$

Firstly, by using the same order σ^*, and posting the same prices $\{p_i^*\}_{i=1}^n$ to the new buyers, the probability of selling to each buyer remains the same and thus, this sequential posted-pricing mechanism extracts the same revenue as $\mathsf{SPM}(\{F_i\}_{i=1}^n)$.

For the second property, let p^* be the optimal anonymous price for the triangular instance, then

$$\begin{aligned}\mathsf{AP}(\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^n) &= \mathsf{AP}(p^*, \{\mathrm{Tri}(v_i, q_i)\}_{i=1}^n) \\ &\leq \mathsf{AP}(p^*, \{F_i\}_{i=1}^n) \\ &\leq \mathsf{AP}(\{F_i\}_{i=1}^n),\end{aligned}$$

where the first inequality follows from the fact that F_i stochastically dominates $\mathrm{Tri}(v_i, q_i)$ for all $i \in [n]$ (illustrated in Figure 4(b)).

[Lemma 3.2]. *Given a triangular instance $\{Tri(v_i, q_i)\}_{i=1}^n$ with $v_1 \geq v_2 \geq \cdots \geq v_n$, (1) the optimal sequential posted-price mechanism posts price $p_i = v_i$ to the i-th buyer, for i from 1 to n; and (2) $\mathsf{SPM} = \mathsf{OPT} = \sum_{i=1}^n v_i q_i \cdot \prod_{j=1}^{i-1} (1 - q_j)$.*

The remark below was proposed by an anonymous reviewer (of an early version of this paper), which would be helpful for understanding Lemma 3.2. The notion of virtual value can be referred in the seminal work of Myerson [46].

Remark 3. *For a buyer with triangular distribution $Tri(v_i, q_i)$, the only positive virtual value is v_i. Hence, for a triangular instance $\{Tri(v_i, q_i)\}_{i=1}^n$, the SPM in Lemma 3.2 states (1) sorting buyers in decreasing order (according to v_i's); then (2) letting each buyer i make take-it-or-leave-it decision, at his (unique) positive virtual value v_i. This is exactly running Myerson Auction.*

Proof. For convenience, we only deal with the case that $v_i < \infty$ for all $i \in [n]$. The case that some v_i's equal to ∞ follows by letting those v_i's go from finite numbers to infinity.

Suppose the seller compels the buyers to come for i from 1 to n, and posts price $p_i = v_i$ to the i-th buyer. In this case, when the i-th buyer comes,

1. The item remains unsold with probability $\prod_{j=1}^{i-1} (1 - q_j)$;
2. If so, the expected revenue from the i-th buyer is $v_i q_i$.

Hence, the seller gains exactly $\sum_{i=1}^n v_i q_i \cdot \prod_{j=1}^{i-1} (1 - q_j)$ from such sequential posted-pricing mechanism. To conquer the lemma, it remains to show that

$\mathsf{OPT} \le \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j)$. By Myerson's seminal work [46], the expected revenue from OPT equals to the expectation of the highest virtual value, that is,

$$\mathsf{OPT} = \int_0^\infty \left(1 - \prod_{i=1}^{n} F_i^{\varphi}(x)\right) dx,$$

where $F_i^{\varphi}(p)$ be the CDF of the i-th buyer's virtual value. For a triangular distribution $\mathrm{Tri}(v_i, q_i)$,

$$p - \frac{1-F_i(p)}{f_i(p)} = p - \frac{1 - \frac{(1-q_i)\cdot p}{(1-q_i)\cdot p + v_i q_i}}{\frac{(1-q_i)\cdot v_i q_i}{[(1-q_i)\cdot p + v_i q_i]^2}} = -\frac{v_i q_i}{1-q_i},$$

for all $p \in (0, v_i)$. Accordingly,

$$F_i^{\varphi}(p) = \begin{cases} 0 & p \in \left(\infty, -\frac{v_i q_i}{1-q_i}\right) \\ 1-q_i & p \in \left[-\frac{v_i q_i}{1-q_i}, v_i\right) \\ 1 & p \in [v_i, \infty) \end{cases}.$$

Define $v_0 \overset{\text{def}}{=} \infty$ and $v_{n+1} \overset{\text{def}}{=} 0$ for notational simplicity. It is easy to see $\prod_{i=1}^{n} F_i^{\varphi}(p) = \prod_{i=1}^{k}(1-q_i)$ for all $p \in [v_{k+1}, v_k)$ and $k \in \{0\} \bigcup [n]$ and thus, $\mathsf{OPT} = \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j)$. This completes the proof of the lemma.

LEMMA A.1. *In a worst-case instance, w.l.o.g.* $v_1 > v_2 > \cdots > v_n \ge 1$.

Proof. Suppose $v_k = v_{k+1}$ for some $1 \le k < n$, we construct a new instance $\{\mathrm{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^{n-1}$ as follow:

$$\overline{v}_i = \begin{cases} v_i, & i \le k \\ v_{i+1}, & k < i \le n-1 \end{cases}$$

$$\overline{q}_i = \begin{cases} q_i, & i \le k-1 \\ q_k + q_{k+1} - q_k q_{k+1}, & i = k \\ q_{i+1}, & k < i \le n-1 \end{cases}.$$

It suffices to prove that:
1. $\mathsf{SPM}\left(\{\mathrm{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^{n-1}\right) = \mathsf{SPM}\left(\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^{n}\right)$;
2. $\mathsf{AP}\left(\{\mathrm{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^{n-1}\right) \le \mathsf{AP}\left(\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^{n}\right)$.

For the first claim, according to Lemma 3.2, $\mathsf{SPM}\left(\{\mathrm{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^{n-1}\right)$ equals to

$$\begin{aligned}
&\sum_{i=1}^{n-1} \overline{v}_i \overline{q}_i \cdot \prod_{j=1}^{i-1}(1-\overline{q}_j) \\
&= \sum_{i=1}^{k-1} \overline{v}_i \overline{q}_i \cdot \prod_{j=1}^{i-1}(1-\overline{q}_j) + \Bigg[\overline{v}_k \overline{q}_k \\
&+ \sum_{i=k+1}^{n-1} \overline{v}_i \overline{q}_i (1-\overline{q}_k) \cdot \prod_{j=k+1}^{i-1}(1-\overline{q}_j)\Bigg] \cdot \prod_{j=1}^{k-1}(1-\overline{q}_j) \\
&\overset{(\diamond)}{=} \sum_{i=1}^{k-1} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j) + \Bigg[v_k q_k + v_{k+1} q_{k+1}(1-q_k) \\
&+ \sum_{i=k+2}^{n} v_i q_i (1-q_k)(1-q_{k+1}) \cdot \prod_{j=k+2}^{i}(1-q_j)\Bigg] \\
&\cdot \prod_{j=1}^{k-1}(1-q_j) \\
&= \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j) = \mathsf{SPM}\left(\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^{n}\right),
\end{aligned}$$

where $(\diamond)$ follows from the fact that $1 - \overline{q}_k = (1-q_k)(1-q_{k+1})$.

To prove the second claim, we only need to prove that $\overline{F}_k(p) \ge F_k(p) \cdot F_{k+1}(p)$ for all p. This is trivial when $p > v_k$, since $\overline{F}_k(p) = F_k(p) = F_{k+1}(p) = 1$. For $p \le v_k$, we have

$$\begin{aligned}
&\left(\overline{F}_k(p)\right)^{-1} - \left(F_k(p) \cdot F_{k+1}(p)\right)^{-1} \\
&= 1 + \frac{\overline{v}_k \overline{q}_k}{1-\overline{q}_k} \cdot \frac{1}{p} - \left(1 + \frac{v_k q_k}{1-q_k} \cdot \frac{1}{p}\right) \\
&\qquad \cdot \left(1 + \frac{v_{k+1} q_{k+1}}{1-q_{k+1}} \cdot \frac{1}{p}\right) \\
&= -\frac{q_k q_{k+1}}{(1-q_k)\cdot(1-q_{k+1})} \cdot \left(\frac{v_k}{p} - 1\right) \cdot \frac{v_k}{p} \le 0,
\end{aligned}$$

which concludes the proof.

LEMMA A.2. *In a worst-case instance, there is a buyer with CDF* $F_0(p) = \frac{p}{p+1}$.

Proof. Observe that $v_i q_i \le 1$ for each $1 \le i \le n$, since

$$\begin{aligned}
1 \ge \mathsf{AP}(v_i) &= v_i \cdot \left(1 - \prod_{j=1}^{n} F_j(v_i)\right) \\
&\ge v_i \cdot (1 - F_i(v_i)) = v_i q_i.
\end{aligned}$$

Besides, we assume w.l.o.g. $\mathsf{SPM} = \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j) > 1$ and define $k \overset{\text{def}}{=} \arg\min_{1\le i\le n} \left\{\sum_{j=1}^{i} v_j q_j > 1\right\}$. Now consider the new instance $\text{Tri}(\overline{v}_i, \overline{q}_i)_{i=1}^{n-k+1}$ below with $(n-k+1)$ buyers,

(A.1)
$$\overline{v}_1 = v_k \qquad \overline{q}_1 = \frac{1}{\overline{v}_1} \cdot \left(\sum_{i=1}^{k} v_i q_i - 1 \right),$$

(A.2)
$$\overline{v}_i = v_{i+k-1} \quad \overline{q}_i = q_{i+k-1} \quad \forall \quad 2 \le i \le n-k+1.$$

We claim that

1. $\{\text{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^{n-k+1}$, together with $\text{Tri}(\infty)$, is feasible to constraint (3.3);
2. $1 + \mathsf{SPM}\left(\{\text{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^{n-k+1}\right) \ge \mathsf{SPM}\left(\{\text{Tri}(v_i, q_i)\}_{i=1}^{n}\right)$

According to the definition of k, we know $2 \le k \le n$, $\sum_{i=1}^{k-1} v_i q_i \le 1$ and $\overline{q}_1 \le q_k$.

We first deal with the feasibility of instance $\{\text{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^{n-k+1}$. The case that $p \in (\overline{v}_1, \infty)$ is trivial, since no one has value above p.

When $p \in (1, \overline{v}_1]$, since for those $i \ge 2$, the new instances are just equal to previous instance with index $(k-1)$ greater, we only need to verify

$$\begin{aligned}
\prod_{i=1}^{k} (F_i(p))^{-1} &= \prod_{i=1}^{k} \left(1 + \frac{v_i q_i}{1-q_i} \cdot \frac{1}{p}\right) \\
&\ge \left(1 + \frac{1}{p}\right) \cdot \left(1 + \frac{\overline{v}_1 \overline{q}_1}{1-\overline{q}_1} \cdot \frac{1}{p}\right) \\
&= \left(\overline{F}_0(p)\right)^{-1} \cdot \left(\overline{F}_1(p)\right)^{-1}.
\end{aligned}$$

Dropping $\left(\frac{1}{1-q_i}\right)$ factors (for each $1 \le i \le k-1$) on the left hand side of this inequality, we have

$$\begin{aligned}
&LHS - RHS \\
&> \prod_{i=1}^{k-1} \left(1 + \frac{v_i q_i}{p}\right) \cdot \left(1 + \frac{v_i q_k}{1-q_k} \cdot \frac{1}{p}\right) - RHS \\
&> \left(1 + \frac{1}{p} \cdot \sum_{i=1}^{k-1} v_i q_i\right) \cdot \left(1 + \frac{v_i q_k}{1-q_k} \cdot \frac{1}{p}\right) - RHS \\
&\overset{(A.1)}{=} \frac{v_k(q_k - \overline{q}_1)}{p} \cdot \left[\frac{1}{(1-q_k)(1-\overline{q}_1)} - 1\right] \\
&\quad + \frac{1}{(1-q_k)p^2} \cdot \left(v_k q_k \cdot \sum_{i=1}^{k-1} v_i q_i - \frac{1-q_k}{1-\overline{q}_1} \cdot \overline{v}_1 \overline{q}_1\right) \\
&> \frac{1}{(1-q_k)p^2} \cdot \left(v_k q_k \cdot \sum_{i=1}^{k-1} v_i q_i - \overline{v}_1 \overline{q}_1\right) \\
&\overset{(A.1)}{=} \frac{1}{(1-q_k)p^2} \cdot \left(1 - \sum_{i=1}^{k-1} v_i q_i\right) \cdot (1 - v_k q_k) \ge 0,
\end{aligned}$$

where the second inequality follows from expanding the multiplication, the third inequality follows since $(1-q_k)(1-\overline{q}_1) < 1$ and $\overline{q}_1 \le q_k$, the last inequality follows from the aforementioned facts that $\sum_{i=1}^{k-1} v_i q_i \le 1$ and $v_k q_k \le 1$. Now constraint (3.3) turns to be

$$\ln\left(1 + \frac{1}{p}\right) + \sum_{i:\overline{v}_i \ge p} \ln\left(1 + \frac{\overline{v}_i \overline{q}_i}{1-\overline{q}_i} \cdot \frac{1}{p}\right) \le \ln\left(\frac{p}{p-1}\right) \qquad \forall p \in (1, \infty),$$

which follows constraint (3.4) by rearranging.

We continue to prove that

$$1 + \mathsf{SPM}\left(\{\text{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^{n-k+1}\right) \ge \mathsf{SPM}\left(\{\text{Tri}(v_i, q_i)\}_{i=1}^{n}\right).$$

By Lemma 3.2, the difference between the left and the right hand side is

$$\begin{aligned}
&1 + \sum_{i=1}^{n-k+1} \overline{v}_i \overline{q}_i \cdot \prod_{j=1}^{i-1} (1-\overline{q}_j) - \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1} (1-q_j) \\
&= \left[1 + \overline{v}_1 \overline{q}_1 - \sum_{i=1}^{k} v_i q_i \cdot \prod_{j=1}^{i-1} (1-q_j)\right] \\
&+ \left[\sum_{i=2}^{n-k+1} \overline{v}_i \overline{q}_i \cdot \prod_{j=1}^{i-1} (1-\overline{q}_j) - \sum_{i=k+1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1} (1-q_j)\right] \\
&\overset{(A.1, A.2)}{=} \sum_{i=1}^{k} v_i q_i \cdot \left[1 - \prod_{j=1}^{i-1} (1-q_j)\right] \\
&+ \sum_{i=k+1}^{n} v_i q_i \cdot \prod_{j=k+1}^{i-1} (1-q_j) \cdot \left[(1-\overline{q}_1) - \prod_{j=1}^{k} (1-q_j)\right] \\
&\quad \ge 0,
\end{aligned}$$

where the inequality follows from the facts that $1 \ge \prod_{j=1}^{i-1} (1-q_j)$, and that $(1-\overline{q}_1) \ge (1-q_k) \ge \prod_{j=1}^{k} (1-q_j)$. This completes the proof of Lemma A.2.

A.2 Missing Proofs in Section 3: Optimal Solution

[Lemma 3.3]. *$\mathcal{R}'(p) = p \cdot \mathcal{Q}'(p)$ and $\mathcal{R}'(p) < \mathcal{Q}'(p) < 0$ when $p \in (1, \infty)$, and*

$$\lim_{p \to \infty} \mathcal{R}(p) = \lim_{p \to \infty} \mathcal{Q}(p) = 0$$
$$\lim_{p \to 1^+} \mathcal{R}(p) = \lim_{p \to 1^+} \mathcal{Q}(p) = \infty.$$

Proof. Since $\mathcal{R} = p \ln\left(\frac{p^2}{p^2-1}\right)$,

$$\mathcal{R}'(p) = -\frac{2}{p^2-1} + \ln\left(1 + \frac{1}{p^2-1}\right) \overset{(\dagger)}{\le} -\frac{1}{p^2-1} < 0,$$

where (†) follows that $\ln(1+x) \le x$ when $x \ge 0$. And since $\mathcal{Q}(p) \overset{\text{def}}{=} \ln\left(\frac{p^2}{p^2-1}\right) - \sum_{k=1}^{\infty} \frac{1}{2k^2} \cdot \frac{1}{p^{2k}}$,

$$\begin{aligned}\mathcal{Q}'(p) &= -\frac{2}{p(p^2-1)} + \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{p^{2k+1}} \\ &= -\frac{2}{p(p^2-1)} - \frac{1}{p} \cdot \ln\left(1 - \frac{1}{p^2}\right) = \frac{\mathcal{R}'(p)}{p},\end{aligned}$$

As per these, clearly $\mathcal{R}'(p) < \mathcal{Q}'(p) < 0$ when $p > 1$. For the first limitation,

$$\begin{aligned}\lim_{p\to\infty} \mathcal{R}(p) &= \lim_{p\to\infty} p \ln\left(1 + \frac{1}{p^2-1}\right) \\ &\le \lim_{p\to\infty} \frac{p}{p^2-1} = 0, \\ \lim_{p\to\infty} \mathcal{Q}(p) &= \lim_{p\to\infty} \int_p^{\infty} (-\mathcal{Q}'(x))\, dx \\ &\le \lim_{p\to\infty} \int_p^{\infty} (-\mathcal{R}'(x))\, dx = \lim_{p\to\infty} \mathcal{R}(p) = 0.\end{aligned}$$

For the second limitation,

$$\begin{aligned}\lim_{p\to 1^+} \mathcal{R}(p) &\ge \ln\left(\lim_{p\to 1^+} \frac{p^2}{p^2-1}\right) = \ln(\infty) = \infty, \\ \lim_{p\to 1^+} \mathcal{Q}(p) &= \lim_{p\to 1^+} \int_p^{\infty} \frac{-\mathcal{R}'(x)}{x} dx \\ &\ge \lim_{p\to 1^+} \int_p^{2} \frac{-\mathcal{R}'(x)}{2} dx \\ &= \lim_{p\to 1^+} \frac{\mathcal{R}(p) - \mathcal{R}(2)}{2} = \infty.\end{aligned}$$

This completes the proof of Lemma 3.3.

[Lemma 3.4]. *Given a triangular instance $\{Tri(v_i, q_i)\}_{i=1}^n$ that constraint (3.5) is not tight for some $k \in [n]$, there exists a triangular instance $\{Tri(\overline{v}_i, \overline{q}_i)\}_{i=1}^n$ such that (1) $\sum_{i=1}^{k} \ln\left(1 + \frac{\overline{v}_i \overline{q}_i}{1-\overline{q}_i}\right) = \mathcal{R}(\overline{v}_k)$, for all $k \in [n]$; and (2) $\sum_{i=1}^{n} \overline{v}_i \overline{q}_i \cdot \prod_{j=1}^{i-1}(1-\overline{q}_j) > \sum_{i=1}^{n} v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j)$.*

Proof. We construct such $\{\text{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^n$ by induction. Assume w.l.o.g. k is the smallest index for constraint (3.5) that is not tight, presented in the following:

(A.3)
$$\sum_{j=1}^{i} \ln\left(1 + \frac{v_j q_j}{1-q_j}\right) = \mathcal{R}(v_i) \qquad \forall i \in [k-1],$$

(A.4)
$$\ln\left(1 + \frac{v_k q_k}{1-q_k}\right) < \mathcal{R}(v_k) - \mathcal{R}(v_{k-1}).$$

We construct a new instance $\{\text{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^n$ in the following way: For each $i \ne k$, let

(A.5)
$$\overline{v}_i = v_i \qquad \overline{q}_i = q_i.$$

For $i = k$, define $\overline{v}_k$ and $\overline{q}_k$ that satisfy

(A.6)
$$\ln\left(1 + \frac{\overline{v}_k \overline{q}_k}{1-\overline{q}_k}\right) = \mathcal{R}(\overline{v}_k) - \mathcal{R}(v_{k-1}) \qquad \overline{v}_k \overline{q}_k = v_k q_k.$$

After such assignment, we claim that

1. Such $\overline{v}_k$ and $\overline{q}_k$ certainly exist, and $v_k < \overline{v}_k < v_{k-1}$ and $0 < \overline{q}_k < q_k$.
2. W.r.t. $\{\text{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^n$, constraint (3.5) strictly holds for each $k+1 \le i \le n$.

For the first claim, define $K(x) \overset{\text{def}}{=} \ln\left(1 + \frac{v_k q_k}{1 - \frac{v_k q_k}{x}}\right) - \mathcal{R}(x) + \mathcal{R}(v_{k-1})$. Observe that

$$K(v_k) = \ln\left(1 + \frac{v_k q_k}{1-q_k}\right) - \mathcal{R}(v_k) + \mathcal{R}(v_{k-1}) \overset{(A.4)}{<} 0$$

$$K(v_{k-1}) = \ln\left(1 + \frac{v_k q_k}{1 - \frac{v_k q_k}{v_{k-1}}}\right) > 0,$$

and that $K(x)$ is continuous when $x \in [v_k, v_{k-1}]$. According to the intermediate value theorem, there exists at least one $\overline{v}_k \in (v_k, v_{k-1})$ such that $K(\overline{v}_k) = 0$ and $\overline{q}_k = \frac{v_k q_k}{\overline{v}_k} < q_k$.

For the second claim,

$$\begin{aligned}&\sum_{j=1}^{i} \ln\left(1 + \frac{\overline{v}_j \overline{q}_j}{1-\overline{q}_j}\right) \\ &\overset{(A.5)}{=} \sum_{j=1}^{i} \ln\left(1 + \frac{v_j q_j}{1-q_j}\right) + \ln\left(1 + \frac{\overline{v}_k \overline{q}_k}{1-\overline{q}_k}\right) \\ &\quad - \ln\left(1 + \frac{v_k q_k}{1-q_k}\right) \\ &\overset{(3.5, A.6)}{\le} \mathcal{R}(v_i) + \ln\left(1 + \frac{v_k q_k}{1-\overline{q}_k}\right) - \ln\left(1 + \frac{v_k q_k}{1-q_k}\right) \\ &< \mathcal{R}(v_i),\end{aligned}$$

for each $k+1 \le i \le n$, where the last inequality is strict since $0 < \overline{q}_k < q_k$.

Combining the above two claims together, for each $k+1 \le i \le n$, we can construct desired $\overline{v}_i$ and $\overline{q}_i$ inductively. In summary,

$$\overline{v}_i\overline{q}_i = v_i q_i \qquad \overline{v}_i \ge v_i \qquad \overline{q}_i \le q_i, \tag{A.7}$$

for each $i \in [n]$, where both inequalities are strict for each $k \le i \le n$. Eventually, the difference between $\mathsf{SPM}\left(\{\mathrm{Tri}(\overline{v}_i, \overline{q}_i)\}_{i=1}^n\right)$ and $\mathsf{SPM}\left(\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^n\right)$ is

$$\begin{aligned}
&\sum_{i=1}^n \overline{v}_i\overline{q}_i \cdot \prod_{j=1}^{i-1}(1-\overline{q}_j) - \sum_{i=1}^n v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j)\\
\overset{(A.7)}{=}&\sum_{i=1}^n v_i q_i \cdot \left[\prod_{j=1}^{i-1}(1-\overline{q}_j) - \prod_{j=1}^{i-1}(1-q_j)\right] > 0,
\end{aligned}$$

where the last inequality is strict since $\overline{q}_i < q_i$ for each $k \le i \le n$. This completes the proof of Lemma 3.4.

[Lemma 3.5]. *For each $i \in [n]$,*

$$\begin{aligned}
&v_i q_i \cdot \prod_{j=1}^{i-1}(1-q_j) - \int_{v_i}^{v_{i-1}} x \cdot (-\mathcal{Q}'(x)) \cdot e^{-\mathcal{Q}(x)} dx\\
\le &\left[\prod_{j=1}^{i-1}(1-q_j) - \prod_{j=1}^{i}(1-q_j)\right]\\
&- \left(e^{-\mathcal{Q}(v_{i-1})} - e^{-\mathcal{Q}(v_i)}\right).
\end{aligned}$$

Proof. Since $e^{-\mathcal{Q}(v_{i-1})} - e^{-\mathcal{Q}(v_i)} = \int_{v_i}^{v_{i-1}} (-\mathcal{Q}'(x)) \cdot e^{-\mathcal{Q}(x)} dx$, we would rearrange this inequality, and turn to prove

$$\begin{aligned}
&\frac{(v_i-1)q_i}{1-q_i} \cdot \prod_{j=1}^{i}(1-q_j)\\
\le &\int_{v_i}^{v_{i-1}} (x-1) \cdot (-\mathcal{Q}'(x)) \cdot e^{-\mathcal{Q}(x)} dx.
\end{aligned}$$

We would separate both of the left and the right hand sides into two parts, and deal with inequalities (A.8,A.9) instead:

$$\frac{(v_i-1)q_i}{1-q_i} \le \int_{v_i}^{v_{i-1}} (x-1) \cdot (-\mathcal{Q}'(x))\, dx, \tag{A.8}$$

$$\prod_{j=1}^{i}(1-q_j) \le e^{-\mathcal{Q}(x)} \qquad \forall x \in [v_i, v_{i-1}]. \tag{A.9}$$

Applying constraint (3.7) and the fact that $\mathcal{Q}'(p) = \frac{\mathcal{R}'(p)}{p}$ to inequality (A.8),

$$\begin{aligned}
LHS \text{ of } (A.8) =&(v_i - 1) \cdot \frac{e^{\mathcal{R}(v_i)-\mathcal{R}(v_{i-1})} - 1}{v_i},\\
RHS \text{ of } (A.8) =&\left(\mathcal{R}(v_i) - \mathcal{R}(v_{i-1})\right)\\
&- \left(\mathcal{Q}(v_i) - \mathcal{Q}(v_{i-1})\right).
\end{aligned}$$

In this form, we can verify inequality (A.8) by Lemma A.3 below. Similarly, applying constraint (3.7), and the facts that $\mathcal{Q}'(p) < 0$ and $\mathcal{Q}(v_0) = \mathcal{Q}(\infty) = 0$ to inequality (A.9),

$$\begin{aligned}
LHS \text{ of } (A.9) =&\prod_{j=1}^{i}\left(1 + \frac{e^{\mathcal{R}(v_j)-\mathcal{R}(v_{j-1})} - 1}{v_j}\right)^{-1},\\
RHS \text{ of } (A.9) \ge&e^{-\mathcal{Q}(v_i)} = \prod_{j=1}^{i}\left(e^{\mathcal{Q}(v_j)-\mathcal{Q}(v_{j-1})}\right)^{-1}.
\end{aligned}$$

By Lemma A.4, $\left(1 + \frac{e^{\mathcal{R}(v_j)-\mathcal{R}(v_{j-1})}-1}{v_j}\right)^{-1} \le \left(e^{\mathcal{Q}(v_j)-\mathcal{Q}(v_{j-1})}\right)^{-1}$ for each $j \in [i]$. Taking product over all $j \in [i]$ immediately implies inequality (A.9). This finishes the proof of the lemma.

LEMMA A.3. *$(x-1)\cdot\frac{e^{\mathcal{R}(x)-\mathcal{R}(y)}-1}{x} \le (\mathcal{R}(x) - \mathcal{R}(y)) - (\mathcal{Q}(x) - \mathcal{Q}(y))$ when $y > x > 1$.*

Proof. Define $G(x,y) \overset{\text{def}}{=} (x-1)\cdot\frac{e^{\mathcal{R}(x)-\mathcal{R}(y)}-1}{x} + \mathcal{R}(y) - \mathcal{R}(x) - \mathcal{Q}(y) + \mathcal{Q}(x)$, we shall prove that

$$\begin{aligned}
\frac{\partial G}{\partial y} =& -\left(1 - \frac{1}{x}\right) \cdot e^{\mathcal{R}(x)-\mathcal{R}(y)} \cdot \mathcal{R}'(y) + \mathcal{R}'(y) - \mathcal{Q}'(y)\\
=&(-\mathcal{R}'(y)) \cdot e^{-\mathcal{R}(y)} \cdot\\
&\left[\left(1 - \frac{1}{x}\right) \cdot e^{\mathcal{R}(x)} - \left(1 - \frac{1}{y}\right) \cdot e^{\mathcal{R}(y)}\right] \le 0,\\
&(\text{Since } \mathcal{Q}'(y) = \frac{\mathcal{R}'(y)}{y})
\end{aligned}$$

when $y > x > 1$. According to Lemma 3.3 that $\mathcal{R}'(y) < 0$, we turn to show that

$$\ln\left(1 - \frac{1}{x}\right) + \mathcal{R}(x) \le \ln\left(1 - \frac{1}{y}\right) + \mathcal{R}(y).$$

Define $g(x) \overset{\text{def}}{=} \ln\left(1 - \frac{1}{x}\right) + \mathcal{R}(x) = \ln\left(1 - \frac{1}{x}\right) + x\ln\left(\frac{x^2}{x^2-1}\right)$,

$$\begin{aligned}
g'(x) = -\ln\left(1 - \frac{1}{x^2}\right) - \frac{1}{x(x+1)} \ge& \frac{1}{x^2} - \frac{1}{x(x+1)}\\
=& \frac{1}{x^2(x+1)} > 0.
\end{aligned}$$

The monotonicity of $g(x)$ completes the proof of Lemma A.3.

LEMMA A.4. $e^{\mathcal{Q}(x)-\mathcal{Q}(y)} \leq 1 + \frac{e^{\mathcal{R}(x)-\mathcal{R}(y)}-1}{x}$ *when* $y > x > 1$.

Proof. Define $H(x,y) \overset{\text{def}}{=} e^{\mathcal{Q}(x)-\mathcal{Q}(y)} - 1 - \frac{e^{\mathcal{R}(x)-\mathcal{R}(y)}-1}{x}$. It suffices to show that $\frac{\partial H}{\partial y}$

$$
\begin{aligned}
&= e^{\mathcal{Q}(x)-\mathcal{Q}(y)} \cdot (-\mathcal{Q}'(y)) - \frac{1}{x} \cdot e^{\mathcal{R}(x)-\mathcal{R}(y)} \cdot (-\mathcal{R}'(y)) \\
&= (-\mathcal{R}'(y)) \cdot \frac{e^{\mathcal{R}(x)-\mathcal{Q}(y)}}{xy} \\
&\quad \cdot \left(x \cdot e^{\mathcal{Q}(x)-\mathcal{R}(x)} - y \cdot e^{\mathcal{Q}(y)-\mathcal{R}(y)}\right) \leq 0, \\
&\quad (\text{Since } \mathcal{Q}'(y) = \frac{\mathcal{R}'(y)}{y})
\end{aligned}
$$

when $y > x > 1$. Observe that $\mathcal{R}'(y) < 0$, we only need to check that

$$x \cdot e^{\mathcal{Q}(x)-\mathcal{R}(x)} \leq y \cdot e^{\mathcal{Q}(y)-\mathcal{R}(y)}.$$

Define $h(x) \overset{\text{def}}{=} x \cdot e^{\mathcal{Q}(x)-\mathcal{R}(x)}$,

$$
\begin{aligned}
h'(x) &= e^{\mathcal{Q}(x)-\mathcal{R}(x)} \cdot (1 + x \cdot \mathcal{Q}'(x) - x \cdot \mathcal{R}'(x)) \\
&= e^{\mathcal{Q}(x)-\mathcal{R}(x)} \cdot [1 + (x-1) \cdot (-\mathcal{R}'(x))] \geq 0, \\
&\quad (\text{Since } \mathcal{Q}'(x) = \frac{\mathcal{R}'(x)}{x})
\end{aligned}
$$

where the inequality follows from the facts that $x > 1$, and that $\mathcal{R}'(x) < 0$. This completes the proof of Lemma A.4.

[Lemma 3.6]. $1 \leq p^* \leq v_n$, *where* $p^* \overset{\text{def}}{=} \mathcal{Q}^{-1}\left(-\sum_{i=1}^{n} \ln(1-q_i)\right)$.

Proof. Due to Lemma 3.3 that $\mathcal{Q}'(p) < 0$, we turn to verify

$$\mathcal{Q}(1) \geq -\sum_{i=1}^{n} \ln(1-q_i) \geq \mathcal{Q}(v_n).$$

The first inequality follows immediately from the fact that $\lim_{p \to 1^+} \mathcal{Q}(p) = \infty$. The second one follows from inequality (A.9), after assigning $i = n$ and taking $\ln(\cdot)$ on both hand sides of the inequality.

A.3 Missing Proofs in Section 3: Numeric Calculations

For the formula in Theorem 3.1, the function involved in the improper integral is integrable. To see this, we need to transform the formula as following:

$$
\begin{aligned}
&1 + \int_1^\infty x \cdot (-\mathcal{Q}'(x)) \cdot e^{-\mathcal{Q}(x)} dx \\
\overset{(*)}{=}& 2 + \int_1^\infty \left(1 - e^{-\mathcal{Q}(x)}\right) dx \\
\overset{(\ddagger)}{=}& 2 + \int_0^1 \frac{1-(1-t^2) \cdot e^{\frac{1}{2}Li_2(t^2)}}{t^2} dt,
\end{aligned}
$$

where $(*)$ follows from integration by parts, and $(\ddagger)$ follows from integration by substitution (let $t = \frac{1}{x} \in [0,1]$). Now, the integrability can be easily inferred from Figure 5(b). Numerical calculation shows that this number is roughly 2.6202.

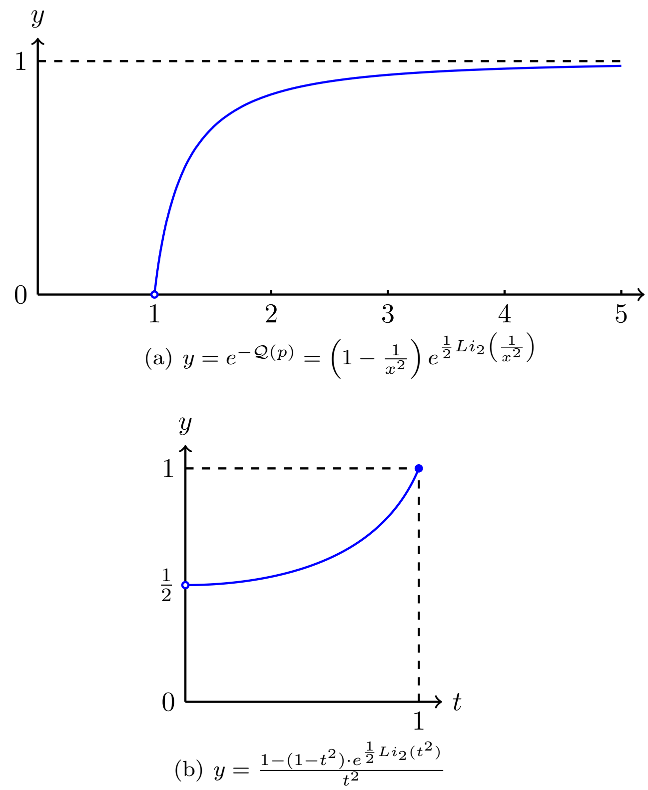


Figure 5: Demonstration for numeric calculation

A.4 Missing Proofs in Section 3: Lower-Bound Analysis

In this part we propose an ϵ-approximate lower-bound instance, Example 2, based on triangular distributions. The feasibility w.r.t constraint (3.4) is relatively easy to deal with, and the ϵ-approximation follows by combining Lemma A.5 and Lemma A.6, and the fact that

$$2 + \int_1^\infty \left(1 - e^{-\mathcal{Q}(x)}\right) dx = 1 + \int_1^\infty x \cdot (-\mathcal{Q}'(x)) \cdot e^{-\mathcal{Q}(x)} dx.$$

EXAMPLE 2. *Given an arbitrarily small* $\epsilon < 1$. *Choose a sufficiently large* $n \in \mathbb{N}_+$, *and define the following triangular instance* $\{Tri(v_i, q_i)\}_{i=1}^{n+1}$:

$$v_i = b - (i-1) \cdot \delta \qquad q_i = \frac{\mathcal{R}(v_i) - \mathcal{R}(v_{i-1})}{v_i + \mathcal{R}(v_i) - \mathcal{R}(v_{i-1})},$$

for each $i \in [n+1]$, *where* $a = \mathcal{Q}^{-1}\left(\ln \frac{8}{\epsilon}\right) > 1$, $b = \frac{8}{\epsilon}$ *and* $\delta = \frac{b-a}{n}$.

The feasibility of this instance w.r.t constraint (3.4) is straightforward:

$$\begin{aligned}&\sum_{i:v_i\geq p}\ln\left(1+\frac{v_iq_i}{1-q_i}\cdot\frac{1}{p}\right)\\&\leq\sum_{i:v_i\geq p}\frac{v_iq_i}{1-q_i}\cdot\frac{1}{p}\\&=\frac{1}{p}\cdot\sum_{i:v_i\geq p}(\mathcal{R}(v_i)-\mathcal{R}(v_{i-1}))\\&\leq\frac{1}{p}\cdot\mathcal{R}(p)=\ln\left(\frac{p^2}{p^2-1}\right).\end{aligned}$$

Where the first inequality holds since $\ln(1+x)\leq x$ when $x\geq 0$), the first equality holds by plug in the formulas for q_i's, and the second inequality holds since $\mathcal{R}(v_0)=0$, $\mathcal{R}(p)$ is decreasing.

LEMMA A.5. *For the triangular instance $\{Tri(v_i,q_i)\}_{i=1}^{n+1}$ in Example 2, with a sufficiently large $n\in\mathbb{N}_+$,*

$$\begin{aligned}&1+\sum_{i=1}^{n+1}v_iq_i\cdot\prod_{j=1}^{i-1}(1-q_j)\\&\geq-\frac{1}{2}\epsilon+1+\int_a^b x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx.\end{aligned}$$

Proof. Plugging the formula for each q_i into the left hand side, we have

$$\begin{aligned}&1+\sum_{i=1}^{n+1}v_iq_i\cdot\prod_{j=1}^{i-1}(1-q_j)\\&=1+v_1q_1+\frac{e^{\mathcal{Q}(v_1)}}{1+\frac{\mathcal{R}(v_1)}{v_1}}\\&\quad\cdot\sum_{i=2}^{n+1}\frac{\mathcal{R}(v_i)-\mathcal{R}(v_{i-1})}{e^{\mathcal{Q}(v_1)}\cdot\prod_{j=2}^{i}\left(1+\frac{\mathcal{R}(v_j)-\mathcal{R}(v_{j-1})}{v_j}\right)}\\&\geq1+e^{\mathcal{Q}(b)-\frac{\mathcal{R}(b)}{b}}\cdot\sum_{i=2}^{n+1}(\mathcal{R}(v_i)-\mathcal{R}(v_{i-1}))\\&\quad\cdot e^{-\left(\mathcal{Q}(b)+\sum_{j=2}^{i}\frac{\mathcal{R}(v_j)-\mathcal{R}(v_{j-1})}{v_j}\right)}.\end{aligned}$$

Where the first inequality holds by dropping v_1q_1, applying facts $v_1=b$ and $1+x\leq e^x$ when $x\geq 0$. Here we apply a standard argument from Riemann integral. Observe that $a=v_{n+1}<v_n<\cdots<v_1=b$ is a uniform partition of interval $[a,b]$, with norm $\delta=\frac{b-a}{n}$. When n approaches to infinity,

$$\lim_{n\to\infty}e^{-\left(\mathcal{Q}(b)+\sum_{j=2}^{i}\frac{\mathcal{R}(v_j)-\mathcal{R}(v_{j-1})}{v_j}\right)}=e^{-\mathcal{Q}(v_i)}\qquad\forall 2\leq i\leq n+1,$$

$$\begin{aligned}&1+\lim_{n\to\infty}\sum_{i=1}^{n+1}v_iq_i\cdot\prod_{j=1}^{i-1}(1-q_j)\\&\geq1+e^{\mathcal{Q}(b)-\frac{\mathcal{R}(b)}{b}}\cdot\int_a^b x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx.\end{aligned}$$

Since $\epsilon>0$ is fixed prior to $n\in\mathbb{N}_+$, we can always choose a sufficiently large $n\in\mathbb{N}_+$ such that

$$\begin{aligned}&1+\sum_{i=1}^{n+1}v_iq_i\cdot\prod_{j=1}^{i-1}(1-q_j)\\&\geq-\frac{1}{4}\epsilon+1+e^{\mathcal{Q}(b)-\frac{\mathcal{R}(b)}{b}}\cdot\int_a^b x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx.\end{aligned}$$

In addition,

$$\begin{aligned}&e^{\mathcal{Q}(b)-\frac{\mathcal{R}(b)}{b}}\geq e^{-\frac{\mathcal{R}(b)}{b}}=1-\frac{1}{64}\epsilon^2>1-\frac{1}{8}\epsilon,\\&\int_a^b x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx\\&\leq\int_1^\infty x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx\approx1.6202<2.\end{aligned}$$

Applying these inequalities, with the sufficiently large $n\in\mathbb{N}_+$ chosen above,

$$1+\sum_{i=1}^{n+1}v_iq_i\cdot\prod_{j=1}^{i-1}(1-q_j)\geq-\frac{1}{2}\epsilon+1+\int_a^b x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx.$$

This concludes our proof of Lemma A.5.

LEMMA A.6. $\left(\int_1^a+\int_b^\infty\right)x\cdot(-\mathcal{Q}'(x))\cdot e^{-\mathcal{Q}(x)}dx\leq\frac{1}{2}\epsilon.$

Proof. Applying integration by part to the left hand side of this inequality,

$$\begin{aligned}LHS=&e^{-\mathcal{Q}(a)}+\int_1^a\left(e^{-\mathcal{Q}(a)}-e^{-\mathcal{Q}(x)}\right)dx\\&+b\cdot\left(1-e^{-\mathcal{Q}(b)}\right)+\int_b^\infty\left(1-e^{-\mathcal{Q}(x)}\right)dx.\end{aligned}$$

Observe that $\mathcal{Q}'(p)<0$, and that $a<\mathcal{Q}^{-1}(\ln 8)\approx 1.0325<2$ (since $\epsilon<1$),

$$\begin{aligned}&e^{-\mathcal{Q}(a)}+\int_1^a\left(e^{-\mathcal{Q}(a)}-e^{-\mathcal{Q}(x)}\right)dx\\&\leq e^{-\mathcal{Q}(a)}+(a-1)\cdot e^{-\mathcal{Q}(a)}\leq2e^{-\mathcal{Q}(a)}=\frac{1}{4}\epsilon.\end{aligned}$$

Besides, $\mathcal{Q}(p)=\ln\left(\frac{p^2}{p^2-1}\right)-\frac{1}{2}Li_2\left(\frac{1}{p^2}\right)\leq\ln\left(\frac{p^2}{p^2-1}\right)$, from which we can infer that $1-e^{-\mathcal{Q}(p)}\leq\frac{1}{p^2}$. Hence,

$$\begin{aligned}&b\cdot\left(1-e^{-\mathcal{Q}(b)}\right)+\int_b^\infty\left(1-e^{-\mathcal{Q}(x)}\right)dx\\&\leq b\cdot\frac{1}{b^2}+\int_b^\infty\frac{1}{x^2}dx=\frac{2}{b}=\frac{1}{4}\epsilon.\end{aligned}$$

Combining everything together completes the proof of Lemma A.6, accordingly the ϵ-approximation of $\{\text{Tri}(v_i, q_i)\}_{i=1}^n$ is achieved.

B Missing Proofs in Section 4

B.1 Proof of Lemma 4.1

[Lemma 4.1 ([17])]. *For any reserve price* $p \in [0, \infty)$, $\mathsf{AR}(p) = p \cdot (1 - D_1(p)) + \int_p^\infty (1 - D_2(x))\, dx$.

Proof. For all $p \in [0, \infty)$, $\mathsf{AR}(p)$ can be decomposed into two parts: 1. A revenue of p when there is at least one agent whose bid is above p; 2. Some extra revenue when there are at least two agents whose bids are above p. The first part is obviously $p \cdot (1 - D_1(p))$. To calculate the second part, we shall notice that, the probability that the seller gain a revenue of $p' > p$, is exactly the probability that there are two or more buyers whose bids are above p', which equals to $(1 - D_2(p'))$. Thus, the second part can be formulated as $\int_p^\infty (1 - D_2(x))\, dx$. Combining these two parts together concludes the proof.

B.2 AR vs. AP: Lower-Bound Analysis in I.I.D. General Setting

We provide a lower-bound example in i.i.d. general setting here, while deferring a more complicated one (with asymmetric regular distributions) to Appendix B.3. Given an arbitrarily small $\epsilon > 0$, we find an instance $\{\overline{F}\}^n$ feasible to Program (P4), and producing a solution no less than $\left(\frac{\pi^2}{6} - \epsilon\right)$. For convenience, we reuse functions $\Phi_1(p)$ and $\Phi_2(p)$ defined in Section 4.

Example 3. (I.I.D. General Setting) *Suppose there are* $n \in \mathbb{N}_+$ *i.i.d. buyers that all follow distribution*

$$\overline{F}(p) = \begin{cases} 0 & p \in [0, 1] \\ \left(1 - \frac{1}{p}\right)^{\frac{1}{n}} & p \in (1, \infty) \end{cases}.$$

For each $n \in \mathbb{N}_+$, we have $\left(\overline{F}(p)\right)^n = \Phi_1(p)$, which implies the feasibility as $p \cdot (1 - \Phi_1(p)) = 1$. And for the ϵ-approximation, when $p \in (1, \infty)$ and n approaches to infinity,

$$\lim_{n\to\infty} \left(\overline{F}(p)\right)^n \cdot \left[1 + n \cdot \left(\frac{1}{\overline{F}(p)} - 1\right)\right]$$
$$= \left(1 - \frac{1}{p}\right) \cdot \left[1 + \lim_{n\to\infty} \frac{e^{-\frac{1}{n} \cdot \ln\left(1 - \frac{1}{p}\right)} - 1}{\frac{1}{n}}\right] = \Phi_2(p).$$

As per this, fix an arbitrarily small $\epsilon > 0$, we can choose a sufficiently large $n \in \mathbb{N}_+$ such that

$$\mathsf{AR}(1) = 1 + \int_1^\infty [1 - \Phi_2(x)]\, dx - \epsilon = \frac{\pi^2}{6} - \epsilon.$$

Remark 4. *For each* $n \geq 2$, *this distribution* $\overline{F}$ *is irregular, since its revenue-quantile curve* $\overline{r}(q) = \frac{1}{1-(1-q)^n}$ *is strictly convex. Noticeably,* $\overline{F}$ *stochastically dominates any other distributions feasible to Program (P4), due to the fact that* $\left(\overline{F}(p)\right)^n = \Phi_1(p)$. *As per these, one can easily see that for a specific* $n \geq 2$, *this* $\overline{F}$ *is still the worst-case distribution. For some typical* n, *we list the tight ratios of* AR *to* AP *in Table 4.*

B.3 AR vs. AP: Lower-Bound Analysis in Asymmetric Regular Setting

In this part we use triangular distributions to construct a lower-bound example in asymmetric regular setting, such that for any given $\epsilon > 0$, the ratio of AR to AP is at least $\left(\frac{\pi^2}{6} - \epsilon\right)$. The key idea here is similar to it involved in Example 2. And for convenience, we define $\mathcal{V}(p) \overset{\text{def}}{=} p \ln\left(\frac{p}{p-1}\right)$.

Example 4. *Given an arbitrarily small* $\epsilon < 1$. *Choose a sufficiently large* $n \in \mathbb{N}_+$, *and define the following triangular instance* $\{\text{Tri}(v_i, q_i)\}_{i=1}^{2n}$:

$$v_i = b \qquad q_i = \frac{\frac{1}{n} \cdot \mathcal{V}(v_i)}{v_i + \frac{1}{n} \cdot \mathcal{V}(v_i)} \qquad \forall 1 \leq i \leq n,$$
$$v_i = b - (i - n) \cdot \delta \qquad q_i = \frac{\mathcal{V}(v_i) - \mathcal{V}(v_{i-1})}{v_i + \mathcal{V}(v_i) - \mathcal{V}(v_{i-1})} \qquad \forall n + 1 \leq i \leq 2n,$$

where $a = \frac{4}{4-\epsilon} > 1$, $b = \frac{4}{\epsilon}$ *and* $\delta = \frac{b-a}{n}$.

The feasibility that $D_1(p) = \prod_{i:v_i \geq p} \left(\frac{p}{p + \frac{v_i q_i}{1-q_i}}\right) \geq \Phi_1(p)$ for all $p \in [0, \infty)$ is equivalent to

$$\ln D_1(p)$$
$$= -\sum_{i:v_i \geq p} \ln\left(1 + \frac{v_i q_i}{1 - q_i} \cdot \frac{1}{p}\right) \geq -\sum_{i:v_i \geq p} \frac{v_i q_i}{1 - q_i} \cdot \frac{1}{p}$$
(Since $\ln(1 + x) \leq x$ when $x \geq 0$)
$$= -\frac{1}{p} \cdot \left[\mathcal{V}(b) \cdot \mathbb{1}_{\leq b}(p) + \sum_{i=n+1}^{v_i \geq p} (\mathcal{V}(v_i) - \mathcal{V}(v_{i-1}))\right]$$
(Plug in the formulas for q_i's)
$$\geq -\frac{1}{p} \cdot \mathcal{V}(p) = \ln \Phi_1(p).$$
(Since $\mathcal{V}(p)$ is decreasing)

Furthermore, the ϵ-approximation is settled in the following lemma.

n	2	3	4	$\cdots$	∞
$(\mathsf{AR}/\mathsf{AP})_n$	$2\ln 2 \approx 1.3863$	$3\ln 3 - \frac{\pi}{\sqrt{3}} \approx 1.4820$	$9\ln 2 - \frac{3\pi}{2} \approx 1.5259$	$\cdots$	$\frac{\pi^2}{6} \approx 1.6449$

Table 4: tight ratio for different n with i.i.d. distributions

LEMMA B.1. *For the triangular instance $\{Tri(v_i, q_i)\}_{i=1}^{2n}$ in Example 4, with a sufficiently large $n \in \mathbb{N}_+$,*

$$\mathit{AR} \geq \mathit{AR}(a) = a \cdot (1 - D_1(p)) + \int_a^b (1 - D_2(x))\, dx \geq \frac{\pi^2}{6} - \epsilon.$$

Proof. Here we apply a standard argument from Riemann integral. Observe that $a = v_{2n} < v_{2n-1} < \cdots < v_{n+1} < v_n = b$ is a uniform partition of interval $[a, b]$, with norm $\delta = \frac{b-a}{n}$. When n approaches to infinity, each q_i tends to 0^+, and for all $p \in [a, b]$,

$$\lim_{n\to\infty} D_1(p) = \exp\left[-\lim_{n\to\infty} \sum_{i:v_i \geq p} \ln\left(1 + \frac{v_i q_i}{1-q_i} \cdot \frac{1}{p}\right)\right]$$
$$= e^{-\frac{\mathcal{V}(b)+(\mathcal{V}(p)-\mathcal{V}(b))}{p}} = \Phi_1(p),$$
$$\lim_{n\to\infty} D_2(p) = \Phi_1(p) \cdot \left(1 + \frac{1}{p} \cdot \lim_{n\to\infty} \sum_{v_i \geq p} \frac{v_i q_i}{1-q_i}\right)$$
$$= \Phi_1(p) \cdot (1 - \ln \Phi_1(p)) = \Phi_2(p).$$

As per these, and since ϵ is fixed prior to $n \in \mathbb{N}_+$, we can always choose a sufficiently large $n \in \mathbb{N}_+$ such that

$$\mathsf{AR}(a) \geq -\frac{1}{2}\epsilon + a \cdot (1 - \Phi_1(a)) + \int_a^b (1 - \Phi_2(x))\, dx$$
$$\overset{(4.10)}{=} -\frac{1}{2}\epsilon + \frac{\pi^2}{6} - \left(\int_1^a + \int_b^\infty\right)(1 - \Phi_2(x))\, dx.$$

Since $1 - \Phi_2(p) = \frac{1}{p} + \left(1 - \frac{1}{p}\right) \cdot \ln\left(1 - \frac{1}{p}\right) \leq \frac{1}{p} - \left(1 - \frac{1}{p}\right) \cdot \frac{1}{p} = \frac{1}{p^2}$,

$$\left(\int_1^a + \int_b^\infty\right)(1 - \Phi_2(x))\, dx \leq \left(\int_1^a + \int_b^\infty\right)\frac{1}{x^2} dx = 1 - \frac{1}{a} + \frac{1}{b} = \frac{1}{2}\epsilon.$$

Combining the above two inequalities together, we complete the proof of Lemma B.1.

C Missing Proofs in Section 5

[Lemma 5.1]. *Given a triangular instance $\{Tri(v_i, q_i)\}_{i=1}^n$ with $v_1 \geq v_2 \cdots \geq v_n > v_{n+1} \overset{def}{=} 0$, the maximum of $\mathsf{AR}(p)$ is achieved by $p = v_i$ for some $i \in [n]$.*

The following remark is due to an anonymous reviewer (of an early version of this paper), and would be of benefit to understanding Lemma 5.1. For this, the notion of virtual value can be referred in the seminal work of Myerson [46].

REMARK 5. *A triangular distribution $Tri(v_i, q_i)$ is only supported on $[0, v_i]$. While a value of v_i corresponds to a virtual value of v_i, any value less than v_i incurs a negative virtual value.*

Given an instance desired by Lemma 5.1. Via increasing the reserve from some $p \in (v_{i+1}, v_i)$ to $p = v_i$, for some $i \in [n]$, the seller no longer allocates the item, when the highest bid is in (v_{i+1}, v_i) (which corresponds to a negative virtual value).

Therefore, $\mathsf{AP}(p)$ raises after such reserve-increase, due to the above virtual-value arguments, and Myerson's specification [46] of the revenue from a truthful mechanism (such as AR).

Proof. We prove the lemma by showing that $\mathsf{AR}(p)$ is non-decreasing in $(v_{i+1}, v_i]$, for all $i \in [n]$ that $v_i > v_{i+1}$. When $p \in (v_{i+1}, v_i]$ for some $1 \leq i \leq n$, plugging the CDF's of $\{\mathrm{Tri}(v_i, q_i)\}_{i=1}^n$ into the formula for $\mathsf{AR}(p)$ from Lemma 4.1,

$$\mathsf{AR}(p) = p \cdot \left[1 - \prod_{j=1}^{i}\left(\frac{p}{p + \frac{v_j q_j}{1-q_j}}\right)\right] + \int_p^\infty \left[1 - \prod_{j:v_j \geq x}\left(\frac{x}{x + \frac{v_j q_j}{1-q_j}}\right) \cdot \left(1 + \frac{1}{x} \cdot \sum_{j:v_j \geq x} \frac{v_j q_j}{1-q_j}\right)\right] dx.$$

We calculate the derivative of $\mathsf{AR}(p)$,

$$\mathsf{AR}'(p) = \prod_{j=1}^{i}\left(\frac{p}{p + \frac{v_j q_j}{1-q_j}}\right) \cdot \sum_{j=1}^{i} \frac{v_j q_j}{1-q_j} \cdot \left(\frac{1}{p} - \frac{1}{p + \frac{v_j q_j}{1-q_j}}\right) > 0,$$

which concludes the lemma.