Liquid Welfare Maximization in Auctions with Multiple Items

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Abstract. Liquid welfare is an alternative efficiency measure for auctions with budget constrained agents. Previous studies focused on auctions of a single (type of) good. In this paper, we initiate the study of general multi-item auctions, obtaining a truthful budget feasible auction with constant approximation ratio of liquid welfare under the assumption of large market.

Our main technique is random sampling. Previously, random sampling was usually used in the setting of single-parameter auctions. When it comes to multi-dimensional settings, this technique meets a number of obstacles and difficulties. In this work, we develop a series of analysis tools and frameworks to overcome these. These tools and frameworks are quite general and they may find applications in other scenarios.

1 Introduction

Let us consider the following auction environment: there is one auctioneer, who has m heterogeneous divisible items and wants to distribute them among n agents. Since the items are divisible, W.L.O.G we assume that each item is of one unit. Each agent i has a value per unit v_{ij} for item j. Each agent i is also constrained by a budget B_i, which is the maximum amount of money i is able to pay during the auction. An allocation rule $A = (x_{ij})_{n\times m}$ specifies the fraction of items everyone is allocated in an auction, where x_{ij} denotes that i is allocated x_{ij} fraction of item j. We say an allocation is feasible if for each item j, $\sum_i x_{ij} \leq 1$. A feasible payment rule $p = (p_1, \ldots, p_n)$ specifies the amount of money each agent needs to pay, while satisfying budget constraint $p_i \leq B_i$. Basically, an auction is an algorithm that takes all agents' bids as inputs, and outputs a feasible allocation and payment rule. We say an auction is *truthful* if it is every agent's dominant strategy to bid her/his true private profile (here it means value and budget). We say an auction is *universally truthful* if this auction is a distribution over deterministic truthful auctions. Put it more precisely, agent i's utility is $x_{ij}v_{ij} - p_i$ if $p_i \leq B_i$ and $-\infty$ if $p_i \geq B_i$. We also assume that the agents are risk-neutral.

Liquid Welfare. Due to the budget constraints, it is impossible to get any reasonable guarantee for the social welfare objective, even in the simplest setting

V. Bilò and M. Flammini (Eds.): SAGT 2017, LNCS 10504, pp. 41–52, 2017.
DOI: 10.1007/978-3-319-66700-3_4

of single item auction. The main obstacle is that we cannot allocate item(s) to an agent with very high value but low budget truthfully. To overcome this, an alternative measure called liquid welfare was proposed in [16]. Basically, each agent's contribution to the liquid welfare is her/his valuation for the allocated bundle, capped by her/his budget. A precise definition is given as follows.

Definition 1. *The liquid welfare of an assignment* $A = (x_{ij})_{n \times m}$ *in the multi-item setting is*

$$LW(A) = \sum_{i=1}^{n} \min\{\sum_{j=1}^{m} v_{ij} x_{ij}, B_i\}.$$

Just like social welfare is the maximum amount of money an omniscient auctioneer can obtain in a budget-free setting, the liquid welfare measure is actually the maximum amount of money an omniscient auctioneer would be able to extract from agents in the budget setting. Therefore, this measure is a quite reasonable efficiency measure in the budget setting. More justification about the liquid welfare can be found in [16]. Our goal is to design a universally truthful, budget feasible mechanism that guarantees some good approximation towards this liquid welfare objective.

For the simplest setting of single item environment, the problem was first studied in [16], where an $O(\log n)$ approximation mechanism was obtained. In a previous work [25], we improved the result to $O(1)$ approximation. Nothing was previously known for multi-item setting. Although the valuation for each item is additive, the total budget for each agent is shared by different items. This fact makes the multi-item setting much more complicated and challenging than single item setting.

Large Market. Generally speaking, the large market assumption says that a single agent's contribution (power) to the total market is very small. There is a number of recent works which are based on this assumption [2,21]. From practical point of view, this is a very realistic assumption especially in the age of internet economy; from theoretical point of view, this assumption is a very interesting mathematical framework to overcome some impossibility results or get better results than general setting.

In this paper, we study the above liquid welfare maximization problem also with the assumption of large market. It is crucial to give a good characterization of this large market assumption. There are a number of alternative definitions characterizing this. We choose the following one:

$$\forall i, B_i \leq \frac{OPT}{m \cdot c},$$

where OPT is the liquid welfare for an optimal allocation and c is some large constant. The quantity of $\frac{OPT}{m}$ represents the average contribution of each item to the total market. Basically, the above assumption says that each agent does not have enough budget to make a significant interference to a typical item in the market.

Results and Techniques. We get the first constant approximation budget feasible truthful mechanism for liquid welfare maximization problem under the large market assumption. Notice that the liquid welfare is an upper bound for revenue obtained in any individual rational auction. From our proof of liquid welfare guarantee by our auction, what we indeed prove is that the revenue from our auction is a constant fraction of the optimal liquid welfare. As a corollary, our mechanism also guarantees a constant approximation in terms of revenue.

The main technique used in designing our auction is random sampling. Random sampling is a very powerful tool in designing truthful mechanisms, which is widely applied in various of different settings [4–7,23,24]. A typical random sampling mechanism follows the following routine: first divides the agents into two groups randomly, then gathers information from one group and uses this information as a guide to design mechanism for the agents in other group. This approach is usually seen to be applied on single item setting. However, for the multi-item setting, there could be a number of equally optimal solutions for the sample set of agents, but the allocations in these different optimal solutions can be quite different for the same item. Such fragility of optimal solutions brings in difficulty in directly applying random sampling: from the sampling set, one can get good estimation of the total welfare of the set, but does not necessary give stable and useful information for individual items. To overcome this, we use a greedy solution rather than the optimal solution as the guidance. The greedy solution has certain robustness and monotonicity properties which are very helpful to get useful information for every single item.

To argue that a random sampling algorithm does give a good guarantee to some objective, the analysis usually has two steps. In the first step, one proves that with a constant probability, the sampling set is a good estimation of the remaining set. Then, in the second step, one proves that under the condition that it is a good sampling, one can get a good allocation from the remaining set. But in the multi-item setting, there are obstacles in proving both steps. With the large market assumption, one can prove that for a single item, with a constant probability, it *is* a good sampling. But to show that a sampling indeed gives a good estimation for a constant fraction of items, it is not sufficient to apply union bound and it is not clear if applying any other tool from probability theory would work, since the correlation between different items could be very complicated. We still do not know how to prove that this is true. Instead, we are able to bypass this with a very subtle and direct estimation of the performance without conditioning that the sampling is a good one. Our analysis has some similarities with that in [12].

Related Work. As budget is becoming an important issue that cannot be neglected in practice, many theoretical investigations have been devoted to analyzing auctions for budget constrained agents. One of the important directions leads to optimal auction design which tries to maximize revenue for the auctioneer [1,8,10,14,20]. Another direction focus on maximizing social efficiency. In particular, there are a number of previous works focusing on a solution concept of Pareto Efficiency [15,22]. Note that the liquid welfare is not the only quantifiable measure for efficiency for budget constrained agents. There are similar alternatives for this measure, studied in [14,27], but for different solution concepts.

Beyond designing auctions that maximize liquid welfare itself, there are other interesting works that follow this liquid welfare notion. For example [3,9,13,18,27] focus on the liquid welfare guarantee at equilibrium. [19] focused on an online version of auctions with budget constraints.

Another line of research is devoted to study budget feasible mechanism design for reversal auction, in which the budget constrained buyer becomes the auctioneer rather than bidder. This model was first proposed and studied by Singer [26]. Since then, several improvements have been obtained [7,11,17].

For random sampling technique applied on mechanism design, there are also a long line of research focusing on it [4–7,23–25]. Most of them are for single item setting. Some of them [4–6] also applied random sampling techniques on multi-item setting. But unlike our setting, they have constraints on solution space, number of agents and value profile.

Open Problems and Discussions. Here, we consider the simplest valuation function, which is linear for each item and additive across different items. It is natural to extend them to more complicated ones. We conjecture that a similar mechanism can be applied to concave (for each item) and sub-modular (across items) functions and leave it to future work.

Theoretically, the most important and interesting open question is whether we can remove the large market assumption and obtain a constant approximation mechanism in general multi item setting. It is easy to see that one can combine the random sampling mechanism with the modified ground bundle second price auction to get an $O(m)$ approximation mechanism. So, it is a constant approximation when the number of items is a constant. But if m is not a constant, the problem remains open.

2 Greedy Algorithm

If all valuations and budgets are common knowledge, the off line liquid welfare maximization problem can be solved by a simple linear program. However, due to the dedication of linear programming, we do not really have much structural understanding or nice properties about this optimal solution. This is in contrast to the single item setting where the optimal solution can be obtained by a simple greedy algorithm.

To overcome this, we propose the following natural greedy algorithm for the multi-item setting. A high level idea of this algorithm is the following: traverse entry (i, j) in decreasing order of value (per unit). At entry (i, j), let agent i buy some fraction of item j at price v_{ij} per unit, so that this fraction is constrained by remaining supply and budget. A detailed formulation can be referred in the following.

Unlike in the single item case, this greedy algorithm is not necessarily optimal but gives a good guarantee towards the optimal. We shall prove that the

Algorithm 1. Greedy Algorithm

```
input  : n agents with valuations (v_ij)_{n×m} and corresponding budgets
         B_1, ..., B_n
output : An allocation (x_ij)_{n×m}
begin
    for each i ∈ [n] do
        C_i ← B_i;
    for each j ∈ [m] do
        s_j ← 1;
    for each i ∈ [n] and j ∈ [m] do
        x_ij ← 0;
    for each v_ij > 0 in decreasing order do
        if C_i > v_ij s_j then
            x_ij ← s_j;
            C_i ← C_i − v_ij s_j;
            s_j ← 0;
        else
            x_ij ← C_i / v_ij;
            s_j ← s_j − C_i / v_ij;
            C_i ← 0;
```

greedy solution is a 2-approximation to optimal liquid welfare, implying that this solution is good enough to serve as a reference to design mechanism. Most importantly, this greedy solution enjoys a number of nice monotonicity properties which are very essential for the analysis of our mechanism in Sect. 3.

In the algorithm, if there are ties among different v_{ij}s, we break them arbitrary but in fixed order (a simple way is to break ties by the index of agents and items). The tie breaking rule gives a total order on v_{ij}'s, thus making the algorithm outputs solution deterministically.

Before we analyze the properties of the algorithm, we introduce a few more necessary notations. Let $A = (x_{ij})_{n\times m}$ be some allocation. We say an allocation is budget compatible if for every i we have $\sum_{j=1}^{m} v_{ij}x_{ij} \leq B_i$. It is obvious that the allocation derived from the above greedy algorithm is budget compatible. For a feasible allocation that is not budget compatible, we can get a new allocation that is budget compatible while achieving the same liquid welfare by just cutting off some fraction of items given to this agent in order to make the value equals to the budget. Thus we can also assume that the optimal allocation given by linear programming is budget compatible. For budget compatible allocations, the liquid welfare is the same as social welfare. For convenience we denote $v_i(A) = \sum_{j=1}^{m} v_{ij}x_{ij}$, $v(A_j) = \sum_{i=1}^{n} v_{ij}x_{ij}$ and $v(A) = \sum_i \sum_j v_{ij}x_{ij}$ respectively. In this paper, unless otherwise specified, we always use $A = (x_{ij})_{n\times m}$ to denote the allocation outputted by the above greedy algorithm. For any subset $T \subseteq [n]$, we use A^T to denote the allocation when running the greedy algorithm only on the subset of agents in T. We use $A^* = (x^*_{ij})_{n\times m}$ to denote a budget compatible optimal allocation.

We first prove that greedy algorithm with full information guarantees at least half of optimal liquid welfare.

Lemma 1. $v(A) \geq \frac{1}{2}OPT$.

Proof. In the greedy algorithm, by decreasing order of v_{ij}s, we always allocate fraction of item j to agent i until agent i's budget is exhausted or item j is sold out. Up to the termination of the algorithm, we denote by $D \subseteq [n]$ the subset of agents who exhaust their budgets ($C_i = 0$), and by $F \subseteq [m]$ the subset of items which are sold out ($s_j = 0$). It is clear that we have $v_{ij} = 0$ if $i \notin D$ and $j \notin F$.

A lower bound of greedy algorithm's liquid welfare is as follows:

$$2v(A) \geq \sum_{i\in D} v_i(A) + \sum_{j\in F} v(A_j) = \sum_{i\in D} B_i + \sum_{j\in F} v(A_j)$$

For i, j such that $i \notin D$ and $j \in F$, we can see that in greedy algorithm, after the algorithm go through this entry (i, j), item j is already sold out. This implies $v_{ij} \leq v(A_j)$ since every fraction of item j is sold at a price of at least v_{ij}.

To bound optimal liquid welfare, we also divide all the agents into two groups: D and the rest. We note that these sets D and F are defined with respect to the greedy solution rather than the optimal solution. We have

$$OPT = v(A^*) = \sum_{i\in D} v_{ij}x^*_{ij} + \sum_{i\notin D} v_{ij}x^*_{ij} = \sum_{i\in D} v_{ij}x^*_{ij} + \sum_{i\notin D}\sum_{j\in F} v_{ij}x^*_{ij},$$

where the last equality uses the fact that $v_{ij} = 0$ for $i \notin D$ and $j \notin F$. We can further bound this by

$$\begin{aligned} OPT = \sum_{i\in D} v_{ij}x^*_{ij} + \sum_{i\notin D}\sum_{j\in F} v_{ij}x^*_{ij} &\leq \sum_{i\in D} B_i + \sum_{i\notin D}\sum_{j\in F} v_{ij}x^*_{ij} \\ &\leq \sum_{i\in D} B_i + \sum_{j\in F} v(A_j) \sum_{i\notin D} x^*_{ij} \\ &\leq \sum_{i\in D} B_i + \sum_{j\in F} v(A_j) \leq 2v(A), \end{aligned}$$

where the first inequality is from the budget compatibility of optimal allocation, the second inequality uses the fact that $v_{ij} \leq v(A_j)$ for $i \notin D$ and $j \in F$ while the third inequality uses that fact that $\sum_{i\notin D} x^*_{ij} \leq 1$ since it is a feasible allocation. This completes the proof. □

Not only greedy algorithm is a good approximation, we shall also prove that it has a nice monotonicity property when running on a subset of the agents. This is crucial for our random sampling mechanism to work.

Lemma 2. (monotonicity of greedy). *Let $T \subseteq [n]$ be a subset of agents, A and A^T be the greedy solutions running on the total set $[n]$ and its subset T respectively. Then $\forall i \in T$ and j, we have $v_i(A^T) \geq v_i(A)$ and $v(A^T_j) \leq v(A_j)$.*

The intuition is clear that when there are less agents, each remaining agent can get more and each item generates less welfare. Notice that this property does not necessary hold for every single agent and item if we use optimal solution rather than greedy solution.

Proof. We prove this by coupling every step of greedy algorithm for the inputs $[n]$ and T. When generating assignments A and A^T, the entries (i, j)s traversed in the algorithm keep the order in v_{ij}, except for $[n]$ it experiences some extra entries (i, j) when $i \notin T$. For these cases, we couple them with empty steps.

For $i \in T$, we denote the remaining budget for agent i by C_i and C_i^T respectively. We also denote remaining supply for item j by s_j and s_j^T respectively. We inductively prove that after each step, $\forall i \in T$ we have $C_i \geq C_i^T$, and $\forall j \in [m]$ we have $s_j \leq s_j^T$.

Initially, $C_i = B_i = C_i^T$ and $s_j = 1 = s_j^T$. Now, we assume that the property holds before the algorithm processes entry (i, j). Notice that after going through an entry (i, j), $\forall k \in T \backslash \{i\}$ both C_k^T and C_k remain the same. Similarly, $\forall l \in [m] \backslash \{j\}$ both s_l^T and s_l remain the same. Thus we only need to consider the changes in C_i, C_i^T, s_j and s_j^T at step (i, j). There are three cases.

$i \notin T$. In this case, the only possible change is that s_j may decrease by some certain amount. So, the monotonicity property remains to hold.

$i \in T$ and $C_i^T \geq s_j^T v_{ij}$. In this case, $C_i \geq C_i^T \geq v_{ij} s_j^T \geq v_{ij} s_j$, thus after step (i, j), $\hat{s}_j^T = \hat{s}_j = 0$, and $\hat{C}_i^T = C_i^T - s_j^T v_{ij} \leq C_i - s_j v_{ij} = \hat{C}_i$, which shows that after this step, the two properties still hold.

$i \in T$ and $C_i^T < s_j^T v_{ij}$. In this case $\hat{C}_i^T = 0 \leq \hat{C}_i$. Also $\hat{s}_j^T = s_j^T - \frac{C_i^T}{v_{ij}} \geq \max\{s_j - \frac{C_i}{v_{ij}}, 0\} = \hat{s}_j$, which shows that after this step, the two properties still hold.

From the above argument, when both algorithms terminate, $\forall i \in T$, $C_i \geq C_i^T$, thus $\forall i \in T$, $v_i(A^T) = B_i - C_i^T \geq B_i - C_i \geq v_i(A)$.

We further prove that $v(A_j^T) \leq v(A_j)$. Since $\forall j$, $s_j \leq s_j^T$ at every step of greedy algorithm, which indicates that each fraction of unit in A is allocated with a price no less than in A^T(if not allocated the price of that fraction is 0). Thus $v(A_j^T) \leq v(A_j)$. □

3 The Random Sampling Mechanism

The greedy algorithm has some nice properties but is obviously not truthful. To design truthful mechanism that has good guarantee, we combine this greedy algorithm with random sampling. The idea is very simple, we randomly divide all the agents into two groups T and R. For agents in T, they do not get any allocation in the auction. We run the greedy algorithm on set T, and use this result as a guide for pricing for agents in R. From the solution of the greedy algorithm, we have a rough idea of how to set the price for each item. Then, we simply sell the items to agents in R at fixed prices which are determined by the output of greedy algorithm. A formal description of the auction is as follows.

We present our main theorem in the following.

Algorithm 2. Random Sampling Mechanism

input : n agents with valuations $(v_{ij})_{n\times m}$ and corresponding budgets $B_1, \ldots, B_n$

output: An allocation and payments

begin

Randomly divide all agents with equal probability into group T and R

$A^T \leftarrow$ the greedy solution running on group T.

for $j \in [m]$ **do**

$p_j = \frac{1}{6}v(A_j^T)$;

Each agent $i \in R$ comes in a given fixed order and buy the most profitable part with respect to price vector $\{p_j\}$ under budget feasibility and unit item supply constraint.

Theorem 1. *The random sampling mechanism is a universal truthful budget feasible mechanism which guarantees a constant fraction of the liquid welfare under the large market assumption.*

The truthfulness and budget feasibility of this auction is obvious. In the following two subsections, we analyse its approximation ratio. Before that, we introduce one more notion: for a subset of agents $T \subseteq [n]$, denote $v(A_j \cap T) = \sum_{i\in T} v_{ij}x_{ij}$.

3.1 Random Sampling and Large Market

We divide all the items into two groups. Let H be the set of easily samplable items consists of item j that satisfies condition $Pr_T(\frac{1}{3}v(A_j) \leq v(A_j \cap T) \leq \frac{2}{3}v(A_j)) \geq \frac{3}{4}$. We also denote the remaining set as G.

Firstly, we provide a simple technical concentration lemma.

Lemma 3. [12] *Let $a_1 \geq a_2 \geq \cdots \geq a_l$ be positive real numbers such that the sum $a = \sum_{i=1}^{l} a_i$ satisfies $a > 36a_1$. We select each number $a_1, \cdots, a_l$ independently at random with probability $1/2$ each and let b be a random variable representing the sum of these selected numbers. Then*

$$Pr[\frac{a}{3} < b < \frac{2a}{3}] \geq \frac{3}{4}.$$

The key lemma in this subsection is as follows. It basically says that the items in group G do not contribute too much in the greedy solution. This is also the only place we use the assumption of large market throughout this paper.

Lemma 4. $\sum_{j\in G} v(A_j) \leq \frac{1}{6}v(A)$

Proof. Lemma 3 provides a sufficient condition for an item to be in H, namely, $B_i \leq \frac{v(A_j)}{36}, \forall i \in [n]$. So, for $j \in G$, there exist $i \in [n]$ such that $B_i > \frac{v(A_j)}{36}$. By the large market assumption, we have that $B_i \leq \frac{OPT}{m\cdot c}$. As a result, we have that $v(A_j) < \frac{36OPT}{m\cdot c} \leq \frac{72v(A)}{m\cdot c}$ for all $j \in G$. Since there are at most m items in G, we get $\sum_{j\in G} v(A_j) \leq \frac{72}{c}v(A)$. By choosing $c = 432$ in the large market assumption, we get the claimed result. □

3.2 Approximation Ratio

In the above section, we already show that items in G do not contribute much. Thus, if our auction do get a good guarantee on items in H, then we are done. In this subsection we will prove this. Before that we introduce a few more necessary definitions. For an item $j \in H$, we denote by Π_j the set of T such that for $T \in \Pi_j$, $\frac{1}{3}v(A_j) \le v(A_j \cap T) \le \frac{2}{3}v(A_j)$. Then, from the definition, we know that for $j \in H$, $Pr(T \in \Pi_j) \ge \frac{3}{4}$. For convenience, we also abuse the notation Π_j to denote the conditional distribution of T over the subset Π_j. We use Π to denote the distribution of T in the mechanism.

The following lemma shows that even if we restrict to the agents in T and only count these T in Π_j, the contribution from items in H is still significant.

Lemma 5.

$$\sum_{j \in H} Pr(T \in \Pi_j) E_{T \sim \Pi_j} v(A_j^T) \ge \frac{1}{8} v(A).$$

Proof. We give both lower bound and upper bound for the term $\sum_j E_{T\sim\Pi} v(A_j^T)$. On one hand, we have

$$\sum_j E_{T\sim\Pi} v(A_j^T) = \sum_i E_{T\sim\Pi} v_i(A^T) \ge \sum_i Pr(i \in T) v_i(A) = \frac{1}{2}\sum_i v_i(A) = \frac{1}{2} v(A),$$

where the inequality uses the fact that $v_i(A^T) \ge v_i(A)$ for any subset T and $i \in T$.

On the other hand, we have

$$\begin{aligned}
\sum_j E_{T\sim\Pi} v(A_j^T) &= \sum_{j\in G} E_{T\sim\Pi} v(A_j^T) + \sum_{j\in H} E_{T\sim\Pi} v(A_j^T) \\
&= \sum_{j\in G} E_{T\sim\Pi} v(A_j^T) + \sum_{j\in H} [Pr(T \in \Pi_j) E_{T\sim\Pi_j} v(A_j^T) \\
&\quad + Pr(T \notin \Pi_j) E_{T\sim\Pi\setminus\Pi_j} v(A_j^T)] \\
&\le \sum_{j\in G} E_{T\sim\Pi} v(A_j) + \sum_{j\in H} [Pr(T \in \Pi_j) E_{T\sim\Pi_j} v(A_j^T) + \frac{1}{4} v(A_j)] \\
&= \frac{1}{4} v(A) + \frac{3}{4}\sum_{j\in G} v(A_j) + \sum_{j\in H} Pr(T \in \Pi_j) E_{T\sim\Pi_j} v(A_j^T) \\
&\le \frac{1}{4} v(A) + \frac{1}{8} v(A) + \sum_{j\in H} Pr(T \in \Pi_j) E_{T\sim\Pi_j} v(A_j^T) \\
&= \frac{3}{8} v(A) + \sum_{j\in H} Pr(T \in \Pi_j) E_{T\sim\Pi_j} v(A_j^T)
\end{aligned}$$

where the first inequality uses the fact that $Pr(T \notin \Pi_j) \le \frac{1}{4}$ and $v(A_j^T) \le v(A_j)$ for all item j, while the last inequality uses Lemma 4.

Connecting the lower and upper bounds for $\sum_j E_{T\sim\Pi} v(A_j^T)$, we have that

$$\sum_{j\in H} Pr(T \in \Pi_j) E_{T\sim\Pi_j} v(A_j^T) \geq \frac{1}{8} v(A).$$

□

Up to this point, we have not talked about the allocation of our random sampling algorithm but only the property of greedy solution under random sampling. The following lemma gives the last piece of the analysis which connects liquid welfare of our mechanism to the above quantity.

Lemma 6. *The liquid welfare obtained from the random sampling algorithm is at least*

$$\frac{1}{12} \sum_{j\in H} Pr(T \in \Pi_j) E_{T\sim\Pi_j} v(A_j^T).$$

Proof. We note that the allocation outputted by our mechanism may not be budget compatible. However, the liquid welfare is always lower bounded by the revenue obtained by a truthful auction (note that the payment of each agent is also bounded by both value and budget), so we only need to bound the revenue obtained by our mechanism.

In our auction, we denote by $D \subseteq R$ the subset of agents who exhaust their budgets, and by $F \subseteq [m]$ the subset of items which are sold out. Both sets are random which depend on the random set T. One key observation is that for all $j \notin F$ and $i \in R \setminus D$, we have $v_{ij} \leq \frac{1}{6} v(A_j^T)$. For $j \notin F$ and $i \in R \setminus D$, agent i did not exhaust his/her budget and item j is not sold out. The only possible reason why agent i did not buy item j is that the price of $\frac{1}{6} v(A_j^T)$ is higher than his/her value v_{ij}, thus we have $v_{ij} \leq \frac{1}{6} v(A_j^T)$.

On one hand, the revenue (and thus the liquid welfare) is bounded by $\sum_{i\in D} B_i$. We further have that

$$\sum_{i\in D} B_i \geq \sum_{i\in D}\sum_{j\notin F} v_{ij}x_{ij} = \sum_{j\notin F}\sum_{i\in D} v_{ij}x_{ij} = \sum_{j\notin F}\left(\sum_{i\in R} v_{ij}x_{ij} - \sum_{i\in R\setminus D} v_{ij}x_{ij}\right)$$
$$\geq \sum_{j\notin F} \max\left\{0, (v(A_j \cap R) - \frac{1}{6} v(A_j^T))\right\}.$$

We note that the allocation A and x_{ij} in the above calculation are from the greedy solution rather than the allocation given by the random sampling mechanism. According to the above argument, this quantity does give a lower bound for the liquid welfare of the random sampling mechanism.

On the other hand, we can also bound the revenue (and thus the liquid welfare) from the item point of view. It is bounded by $\sum_{j\in F} \frac{1}{6} v(A_j^T)$ as the item $j \in F$ is sold out at a price $\frac{1}{6} v(A_j^T)$ per unit.

Let Y_j be a random variable that

$$Y_j := \mathbb{1}_{j\in F} \frac{v(A_j^T)}{6} + \mathbb{1}_{j\notin F} \max\left\{0, (v(A_j \cap R) - \frac{1}{6} v(A_j^T))\right\}.$$

Then the above argument showed that the expected liquid welfare of our mechanism is bounded by $\frac{1}{2}\sum_j E_{T\sim\Pi} Y_j$. We further have that

$$\sum_j E_{T\sim\Pi} Y_j \geq \sum_{j\in H} E_{T\sim\Pi} Y_j \geq \sum_{j\in H} Pr(T \in \Pi_j) E_{T\sim\Pi_j} Y_j,$$

where the inequalities use the simple fact that $Y_j \geq 0$ and one simply throw away some terms in the summation for computing expectation.

For $j \in H$ and $T \in \Pi_j$, we have a better bound for Y_j. If $j \in F$, we directly have $Y_j \geq \frac{1}{6}v(A_j^T)$. For $j \notin F$ we have $Y_j \geq v(A_j \cap R) - \frac{1}{6}v(A_j^T)$. Since $j \in H$ and $T \in \Pi_j$, we have

$$v(A_j \cap R) - \frac{1}{6}v(A_j^T) \geq \frac{1}{3}v(A_j) - \frac{1}{6}v(A_j^T) \geq \frac{1}{3}v(A_j^T) - \frac{1}{6}v(A_j^T) = \frac{1}{6}v(A_j^T).$$

Thus $\forall j \in H$ and $T \in \Pi_j$, we have $Y_j \geq \frac{1}{6}v(A_j^T)$. Therefore, the expected liquid welfare obtained by our mechanism is at least $\frac{1}{12}\sum_{j\in H} Pr(T \in \Pi_j) E_{T\sim\Pi_j} v(A_j^T)$. This completes the proof. □

Put Lemmas 1, 5 and 6 together, we know that the liquid welfare of random sampling mechanism is at least $\frac{1}{192}$ of the optimal one. This completes the proof of the main theorem.

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